Vokasi Unesa Bulletin of Engineering, Technology and Applied Science (VUBETA)
https://journal.unesa.ac.id/index.php/vubeta
Vol. 3, No. 2, 2026, pp. 355~371 DOI: 10.26740/vubeta.v3i2.47699
ISSN: 3064-0768

A Unified Framework for Hyperbolic and Ambiguous Spaces

Anoop Kumar Tiwari[1]
[1]Department of Computer Science and Information Technology, Central University of Haryana, Mahendergarh, India

Article Info

Article history:

Received November 30, 2025
Revised January 24, 2026
Accepted February 15, 2026

Keywords:

Hyperbolic space
Gyrovector space
Ambiguous set
Ambiguous space

ABSTRACT

The concept of space has been a fundamental aspect of mathematical and computational modeling, with Euclidean and hyperbolic spaces serving as classical frameworks. This article explores the transition from hyperbolic space to ambiguous space, establishing a novel comparative framework that integrates gyrovector spaces and ambiguous set theory. Hyperbolic space, characterized by non-Euclidean geometry, forms the foundation for many applications, including machine learning and network analysis. Gyrovector space, an extension of vector space under Möbius addition, provides a computationally efficient model for hyperbolic geometry. In contrast, ambiguous sets introduce a four-dimensional membership structure, enabling more nuanced representations of uncertainty and vagueness in decision-making contexts. The concept of ambiguous space is then developed as a generalized mathematical structure that incorporates elements from both hyperbolic geometry and ambiguous set theory. Finally, we demonstrate the applicability of ambiguous space in customer segmentation, where traditional clustering methods often fail to capture complex consumer behavior.

1. INTRODUCTION

Hyperbolic space (or geometry), a type of non-Euclidean geometry, is characterized by a constant negative curvature. Unlike in Euclidean geometry, the parallel postulate does not hold in hyperbolic geometry, leading to unique properties such as the divergence of parallel lines and exponential area growth [1]. Milnor [2] provided a historical overview of hyperbolic geometry's evolution for the first 150 Years, emphasizing the revolutionary shift it caused in mathematical thought. Stillwell [3] explored the mathematical and philosophical challenges that shaped its early development, where hyperbolic geometry was overlooked in favor of Euclidean and spherical geometries. Iversen [4] delved into the classification of complete hyperbolic surfaces and the uniqueness of hyperbolic planes. Reynolds [5] discussed the hyperboloid model, emphasizing its connections to Minkowski space. Krioukov et al. [6] pointed out that hyperbolic geometry underpins many large-scale networks, providing insights into their hierarchical structures and scalability. In the literature [7]-[10] various kinds of spaces are available, which are summarized below:

- Hyperbolic space (Poincaré disk model): A non-Euclidean space with constant negative curvature, where geodesics are represented as arcs of circles orthogonal to the disk boundary.
- Curvature and manifold space: A mathematical space characterized by varying curvature, often modeled as smooth manifolds with applications in differential geometry.
- Thomas space: A relativistic space influenced by Thomas precession, highlighting rotational effects

*Corresponding Author
Email: dranoop@cuh.ac.in

in special relativity.

- Hilbert space: An infinite-dimensional vector space equipped with an inner product, foundational in quantum mechanics and functional analysis.
- M¨obius gyrovector space (gyrovector space): A mathematical structure that extends vector addition in hyperbolic geometry, enabling efficient modeling of relativistic velocity addition.
- Einstein gyrovector space: A variant of gyrovector space derived from Einstein's theory of relativity, emphasizing relativistic transformations.
- Banach space (analytical space): A complete normed vector space, widely applied in functional analysis for solving differential and integral equations.
- Complex space: A vector space over the field of complex numbers, foundational in quantum physics and signal processing.
- Matrix space (topological space): A space of matrices, often studied with topological structures, enabling analysis of continuity and convergence properties.
- Riemann matrix space: A space of matrices equipped with a Riemannian metric, facilitating applications in geometry and optimization.
- Inner product space: A vector space with an inner product, enabling definitions of angle and length, generalizing Euclidean concepts.
- Vector space: A fundamental algebraic structure consisting of vectors, closed under addition and scalar multiplication.
- Euclidean space: The classical geometry of flat, n-dimensional space, governed by Euclidean axioms and familiar in physical and geometric contexts.

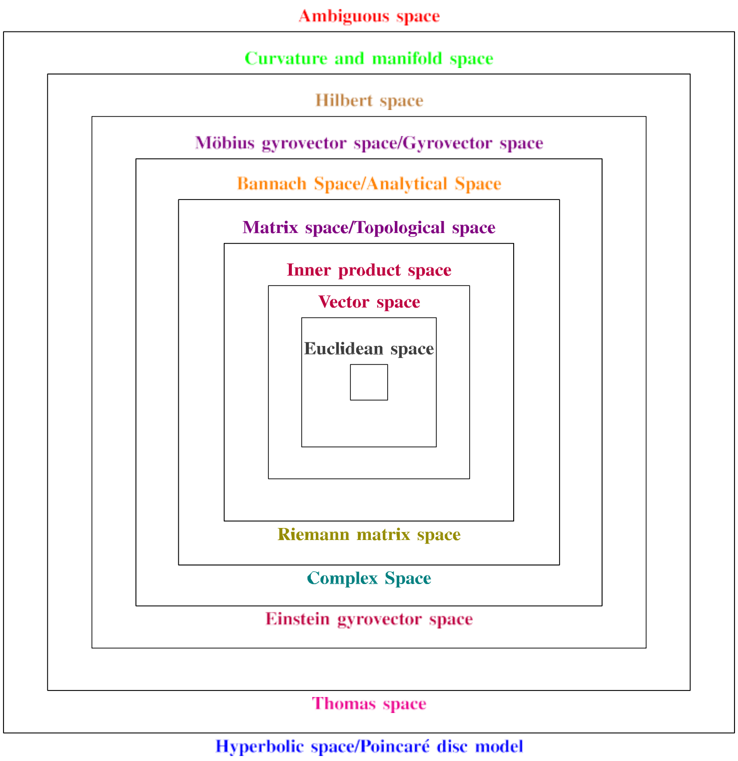

Figure 1. Various types of spaces ranging from fundamental Euclidean and vector spaces to advanced structures such as hyperbolic spaces, Hilbert spaces, and gyrovector spaces. These spaces encompass diverse geometrical, algebraic, and analytical properties with applications across mathematics and physics.

This study introduces ambiguous space, a generalized space that includes uncertain or imprecise definitions and classifications, often used in contexts requiring multiple membership degrees or ambiguous boundaries. All these spaces, including the ambiguous space, are illustrated in Fig. 1, along with their evaluations.

This article is organized as follows. Section 2 discusses the properties and mathematical foundation of hyperbolic space. Section 3 explains the concept of gyrovector space, its operations, and its relation with hyperbolic space. Section 4 defines ambiguous sets and their membership degrees. Section 5 introduces the concept of ambiguous space, its properties, and operations such as addition, multiplication, and scalar multiplication. Section 6 focuses on a specific application of ambiguous space in customer segmentation. Finally, section 7 summarizes the key findings from the article and suggests areas for future research.

1. HYPERBOLIC SPACE

The 2-dimensional *hyperbolic space*, denoted by H^2, is known as the hyperbolic plane [11]-[18]. It is defined by:

Definition 1 (Hyperbolic space). The hyperbolic space of dimension N is a uniquely defined, simply connected N -dimensional Riemannian manifold with a constant sectional curvature of −1. The uniqueness means that any two Riemannian manifolds that satisfy these properties are isometric. This is a consequence of the Killing-Hopf theorem. The following are the properties of hyperbolic space:

a) Hyperbolic geometry has negative constant curvature.
b) Hyperbolic geometry allows parallel lines to intersect.
c) These two properties together allow hyperbolic geometry to be larger than Euclidean geometry.
d) The distance from the origin grows exponentially as we approach the outer disk.

Here is a numerical example of hyperbolic geometry in two-dimensional space, represented using the Poincaré disk model, a common model of 2-dimensional hyperbolic space. Example 1. Consider a point P in the 2-dimensional hyperbolic space H^2 represented by the Poincaré disk. This disk consists of all points inside the unit circle in R^2, where the distance between points is measured differently from the Euclidean distance. The hyperbolic distance $d_h(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the Poincaré disk is given by the formula:

$$d_h(P_1, P_2)\text{arcosh} = \left(1 + \frac{|z_1 - z_2|^2}{2(1-|z_1|^2)(1-|z_2|^2)}\right) \tag{1}$$

where, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are complex representations of the points P_1 and P_2. Consider two points in the Poincaré disk model of H^2

$$P_1 = (0,0), P_2 = \left(\frac{1}{2}, 0\right) \tag{2}$$

The hyperbolic distance $d_h(P_1, P_2)$ between these points is given by the formula:

$$d_h(P_1, P_2)\text{arcosh} = \left(1 + \frac{|z_1 - z_2|^2}{2(1-|z_1|^2)(1-|z_2|^2)}\right) \tag{3}$$

where, $z_1 = 0 + i0$ and $z_2 = \frac{1}{2} + i0$ are the complex representations of the points. Now, calculate the Euclidean distance between P_1 and P_2:

$$|z_1 - z_2|^2 = \left|0 - \frac{1}{2}\right|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \tag{4}$$

Now, compute the denominators:

$$1 - |z_1|^2 = 1 - 0^2 = 1, 1 - |z_2|^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} \tag{5}$$

Thus, the hyperbolic distance is:

$$d_h(P_1, P_2) = arcosh\left(1 + \frac{\frac{1}{4}}{2\times1\times\frac{3}{4}}\right) = arcosh\left(1 + \frac{1}{6}\right) = arcosh\left(\frac{7}{6}\right) \tag{6}$$

Using the inverse hyperbolic cosine function :

$$d_h(P_1, P_2) \approx 0{,}4263 \tag{7}$$

This is the hyperbolic distance between P_1 and P_2 in the Poincaré disk model. The points $P_1 = (0, 0)$ and $P_2 = \frac{1}{2}, 0$ are represented in the unit disk, where the Euclidean distance would be $\frac{1}{2}$, but the hyperbolic distance is larger due to the curvature of the space.

2. GYROVECTOR SPACE

The concept of gyrovector space is a generalization of vector spaces to hyperbolic geometry and was introduced by Abraham A. Ungar [19]. The algebraic framework for computations in hyperbolic spaces is similar to that of vector spaces in Euclidean geometry. The operations that involve gyrovector space are known as "Gyrooperations", and they are defined as:

- Gyroaddition (+): Combines two points while respecting the constraints of hyperbolic geometry.
- Gyrovector scalar multiplication: Scaling operates within the curvature framework.

Let $(G, +)$ be a groupoid, and a, b, c, x ∈ G, then (G, +) is said to be a gyrogroup (denoted by gyr) if the following properties are satisfied. All the properties are discussed by considering $(G, +)$ as a groupoid, where $G = \{0, 1, 2, 3\}$, and + is the binary operation.

Property 1. If $a + b = a + c \Rightarrow b = c$ (left cancellation law).
Proof. Define the binary operation + as:

$$a + b \equiv (a + b)\ mod\ 4, \forall a, b \in G \tag{8}$$

where mod 5 ensures the result wraps around after reaching 5. Now, we aim to verify the left cancellation law, which states:

$$a + b = a + c \Rightarrow b = c, \forall a, b, c \in G \tag{9}$$

- Case 1: Let $a = 1$, $b = 3$, and $c = 3$:

$$a + b = (1 + 3) \equiv 0\ mod\ 4, a + c = (1 + 3) \equiv 0\ mod\ 4$$

Since, $a + b = a + c$, the left cancellation law implies $b = c = 3$.

- Case 2: Let $a = 3$, $b = 1$, and $c = 1$:

$$a + b = (3 + 1) \equiv 0\ mod\ 4, a + c = (3 + 1) \equiv 0\ mod\ 4$$

Again, $a + b = a + c$, so the left cancellation law implies $b = c = 1$.

- Case 3: Let $a = 3$, $b = 2$, and $c = 3$:

$$a + b = (3 + 2) \equiv 1\ mod\ 4, a + c = (3 + 3) \equiv 2\ mod\ 4$$

Here, $a + b /= a + c$, so the left cancellation law does not apply unless $b = c$.

Property 2. If gyr$[0, a] = \boldsymbol{I}$ for the left identity 0 in G. Proof. Define the binary operation + as:

$$gyr[0, a] \equiv I\ mod\ 4, \forall 0, a \in G \tag{10}$$

where mod 5 ensures the result wraps around after reaching 5. Here,

- 0 is the left identity element in the groupoid.
- gyr$[0, a] = \boldsymbol{I}$, where $\boldsymbol{I}$ denotes the *identity transformation*. It implies that applying the gyrogroup operation gyr$[0, a]$ to any element a result in the identity element $\boldsymbol{I}$.

We verify the condition gyr$[0, a] = \boldsymbol{I}$ by showing that for all $a \in G$:

$$gyr[0, a](x) = x \tag{11}$$

In a gyrogroup, the gyration gyr$[0, a](x)$ captures a "twisting" effect of the operation. If gyr$[0, a] = I$, it means that x is unchanged when acted upon, gyr$[0, a](x) = x$. Following is an example for verification

- Let $a = 3, x = 1$: In this context, gyr[0, 3] represents the gyrogroup operation applied to the element 3 (since 0 is the left identity element). Apply this to the equation gyr[0, 3]1 = 1. Since gyr$[0, 3] = I$, we substitute this into the equation:

  $\boldsymbol{I} \cdot 1 = 1$

Thus, gyr[0, 3]1 = 1, as required. This result follows directly from the condition gyr$[0, a] = I$, and the property of the identity element in the groupoid.

- Let $a = 4, x = 2$: In this context, gyr[0, 4] represents the gyrogroup operation applied to the element 4 (since 0 is the left identity element). Apply this to the equation gyr[0, 4]2 = 2. Since gyr[0, 4] = I, we substitute this into the equation:

$$I \cdot 2 = 2$$

Thus, gyr[0, 4]2 = 2, as required. This result follows directly from the condition gyr[0, a] = I, and the property of the identity element in the groupoid.

- Validation for all x:

Since the gyrogroup operation does not modify x, gyr[0, a] = I implies the identity transformation for all $x \in G$.

Remark 1. The identity transformation I is an operation that leaves elements unchanged when it is applied to them. In mathematical terms, for any element x, applying I results in:

$$I \cdot x = x \quad (12)$$

1) In a vector space, the identity transformation is the linear transformation that maps each vector to itself. For any vector v, we have:

$$I \cdot v = v \quad (13)$$

For example, if $v = (2, 3)$, the identity transformation I applied to v gives:

$$I(2,3) = (2,3) \quad (14)$$

2) In group theory, the identity element e of a group G is the element that, when combined with any element of the group, leaves that element unchanged. For a group G with the operation +, the identity element satisfies:

$$e + x = x \text{ and } x + e = x \text{ forall } x \in G \quad (15)$$

For example, in the group of integers (Z, +), the identity element is 0, because:

$$0 + x = x \text{ for all } x \in Z \quad (16)$$

Property 3. If gyr[x, a] = I for any left inverse x of a in G. Proof. In the context of a gyrogroup, gyr[x, a] represents a gyrogroup operation, which is a transformation (or operation) that modifies elements of the group G in a specific way. Intuitively, it measures the “twist” introduced by the elements x and a in the group operation. I denote the identity transformation, which is a special transformation that leaves all elements unchanged. The condition gyr[x, a] = I mean that the gyrogroup operation gyr[x, a] does not twist or change any element of G. Mathematically, this is expressed as:

$$gyr[x, a](c) = c, c \in G \quad (17)$$

An element $x \in G$ is called a left inverse of $a \in G$ if:

$$x + a = 0 \quad (18)$$

This means that when x is combined with a (from the left), the result is the identity element 0. In a gyrogroup, the operation + behaves differently (with a “twist”), but the definition of the left inverse still holds:

$$x + a = 0 \quad (19)$$

Here, x is the element that “cancels” a under the operation +.

For instance, if $x = -a$ works in the gyrogroup (depending on the operation), x would be the left inverse of a. Additionally, the gyrogroup operation property:

$$gyr[x, a] = I \quad (20)$$

may hold, indicating no “twist” occurs. Now, consider the following example and define the binary operation + as:

$$x + a \equiv (x + a) \text{ mod } 4, \forall x, a \in G \quad (21)$$

We are given:

$$a = 3, x = 1(left\ inverse\ of\ a) \quad (22)$$

The left inverse x satisfies $x + a \equiv 0 \mod 4$. For $a = 3$ and $x = 1$:

$$1 + 3 = 4 \equiv 0\ mod 4 \quad (23)$$

Hence, $x = 1$ is the left inverse of $a = 3$.

Property 4. gyr$[a, a] = I$. Proof. Define the binary operation + as:

$$x + a \equiv (x + a)\ mod\ 4 \quad (24)$$

The gyrogroup operation gyr$[a, b]$ must satisfy the condition:

$$gyr[a, a] = I, \quad (25)$$

where I denotes the identity transformation. This means that:

$$gyr[a, a](c) = c\ for\ all\ c \in G \quad (26)$$

We can verify this for $a = 2$ as:

- For $c = 0$:

gyr = 0

- For $c = 1$:

gyr = 1

- For $c = 2$:

gyr = 2

- For $c = 3$:

gyr = 3

In all cases, gyr$[2, 2](c) = c$. This confirms that gyr$[a, a] = I$ holds for $a = 2$ and all $c \in G$.

Property 5. There is a left identity that is also a right identity.

Proof. Define the binary operation + as:

$$x + a \equiv (x + a)\ mod\ 4 \quad (27)$$

The left and right identities can be defined as:

- A left identity e satisfies $e + a = a$ for all $a \in G$.
- A right identity e satisfies $a + e = a$ for all $a \in G$.

Table 1. Computation of $e + a$ and $a + e$ for $e = 0$

a	$0 + a$			$a + 0$		
0	$(0 + 0) \equiv 0$	mod	4	$(0 + 0) \equiv 0$	mod	4
1	$(0 + 1) \equiv 1$	mod	4	$(1 + 0) \equiv 1$	mod	4
2	$(0 + 2) \equiv 2$	mod	4	$(2 + 0) \equiv 2$	mod	4
3	$(0 + 3) \equiv 3$	mod	4	$(3 + 0) \equiv 3$	mod	4

- If e is both a left and right identity, it is simply called the *identity element* of G. To verify the identity element in G, check if there exists an element $e \in G$ such that:

$$e + a = a + e = a, \forall a \in G \quad (28)$$

.

Here, the operation + is defined as:

$$(x + a) \, mod \, 4 \tag{29}$$

Let $e = 0$. Check whether e satisfies the identity conditions for all $a \in G$:

- Left identity: $0 + a \equiv (0 + a) \mod 4 \equiv a$. This holds for all $a \in G$.
- Right identity: $a + 0 \equiv (a + 0) \mod 4 \equiv a$. This also holds for all $a \in G$. Since $0 + a = a + 0 = a$ for all $a \in G$, 0 is both a left and right identity.

For $G = \{0, 1, 2, 3\}$, compute $e + a$ and $a + e$ for $e = 0$ and each $a \in G$ as shown in Table 1. From Table 1, we see that $0 + a = a + 0 = a$ for all $a \in G$. Thus, 0 is the identity element of the groupoid $(G, +)$. In the groupoid $G = \{0, 1, 2, 3\}$ with addition modulo 4, 0 serves as both the left and right identity. Hence, G satisfies the condition that the left identity is also the right identity.

Property 6. Every left inverse is a right inverse. Proof. Left inverse and right inverse can be defined as:

$$x + a = e; \; (Left \; Inverse) \tag{30}$$

$$a + x = e; \; (Right \; Inverse) \tag{31}$$

where, e is the identity element of G. Now, if x satisfies both conditions $x + a = e$ and $a + x = e$, then x is both a left and right inverse of a. From the previous example, we know that 0 is the identity element for $G = \{0, 1, 2, 3\}$, where addition is modulo 4. We calculate both left and right inverses for each $a \in G$ as:

- For $a = 0$:

Left inverse: $x + 0 \equiv 0 \pmod 4 \Rightarrow x = 0$

Right inverse: $0 + x \equiv 0 \pmod 4 \Rightarrow x = 0$ Hence, $x = 0$ is both the left and right inverse of $a = 0$.

- For $a = 1$:

Left inverse: $x + 1 \equiv 0 \pmod 4 \Rightarrow x = 3$

Right inverse: $1 + x \equiv 0 \pmod 4 \Rightarrow x = 3$ Hence, $x = 3$ is both the left and right inverse of $a = 1$.

- For $a = 2$:

Left inverse: $x + 2 \equiv 0 \pmod 4 \Rightarrow x = 2$

Right inverse: $2 + x \equiv 0 \pmod 4 \Rightarrow x = 2$ Hence, $x = 2$ is both the left and right inverse of $a = 2$.

- For $a = 3$:

Left inverse: $x + 3 \equiv 0 \pmod 4 \Rightarrow x = 1$

Right inverse: $3 + x \equiv 0 \pmod 4 \Rightarrow x = 1$ Hence, $x = 1$ is both the left and right inverse of $a = 3$.
It is evident that in the groupoid $G = \{0, 1, 2, 3\}$ with addition modulo 4, every left inverse is also a right inverse.
This property holds for all elements in G.

Property 7. There is only one left identity. Proof. An element $e \in G$ is a left identity if for all $a \in G$:

$$e + a = a \tag{32}$$

A groupoid has only one left identity if no other element besides e satisfies the above condition for all a. We used a groupoid $G = \{0, 1, 2, 3\}$ with addition modulo 4 as the operation:

$$x + a \equiv (x + a) \, mod \, 4 \tag{33}$$

Now, verify each element of $G = \{0, 1, 2, 3\}$ to see if it satisfies $e + a = a$ for all $a \in G$ as:

- $e = 0$:

$$0 + a \equiv (0 + a) \equiv a \, mod \, 4 \tag{34}$$

which holds for all $a \in G$.

- $e = 1, 2, 3$: These fail to satisfy $e + a = a$ for all a. For example,

$$For\ e = 1: 1 + 0 = 1 = 0. \tag{35}$$

Hence, $e = 0$ is the unique left identity in G. In the groupoid $(G, +)$ under addition modulo 4, $e = 0$ is the only element that serves as a left identity. This satisfies the gyrogroup condition that there exists only one left identity.

Property 8. There is only one left inverse of a. Proof. We will use the set $G = \{0, 1, 2, 3\}$ with the addition operation modulo 4, which forms a groupoid. That is, we define the binary operation $+$ on G as:

$$\begin{array}{llll} 0+0=0, & 0+1=1, & 0+2=2, & 0+3=3, \\ 1+0=1, & 1+1=2, & 1+2=3, & 1+3=0, \\ 2+0=2, & 2+1=3, & 2+2=0, & 2+3=1, \\ 3+0=3, & 3+1=0, & 3+2=1, & 3+3=2. \end{array}$$

We can verify the groupoid structure as:

- Closure: The result of adding any two elements of G (modulo 4) is always an element of G.
- Associativity: The operation $+$ is associative in this case because addition modulo 4 satisfies the associative property.
- Existence of identity element: The identity element is 0, since for any element $a \in G$, $a + 0 = a$.
- Left inverses: For each element $a \in G$, we must find a left inverse. A left inverse of a is an element $b \in G$ such that $b + a = 0$.

Now, we can check the left inverse for each element in G as:

1) Left inverse of 0: we need b + 0 = 0. From the above table, we see that 0 + 0 = 0, so the left inverse of 0 is 0
2) Left inverse 1: We need b+1=0. From the above table, we see that 3+1=0, so the left inverse of 1 is 3.
3) Left inverse of 2: We need $b + 2 = 0$. From the table, we see that $2 + 2 = 0$, so the left inverse of 2 is 2.
4) Left inverse of 3: We need $b + 3 = 0$. From the table, we see that $1 + 3 = 0$, so the left inverse of 3 is 1.

From the calculations above, we observe the following left inverses:

- The left inverse of 0 is 0.
- The left inverse of 1 is 3.
- The left inverse of 2 is 2.
- The left inverse of 3 is 1.

Each element has a unique left inverse, and no element has more than one left inverse. Thus, we have demonstrated, with a numerical example, that the set $G = \{0, 1, 2, 3\}$ with addition modulo 4 forms a groupoid and satisfies the condition that each element has a unique left inverse. This satisfies the definition of a gyrogroup, and we have

Property 9. $-a + (a + b) = (-a + a) + b = 0 + b = b$ (left cancellation law). Proof. Consider the set $G = \{0, 1, 2, 3\}$ with the binary operation $+$ defined as addition modulo 4:

$$\begin{array}{llll} 0+0=0, & 0+1=1, & 0+2=2, & 0+3=3, \\ 1+0=1, & 1+1=2, & 1+2=3, & 1+3=0, \\ 2+0=2, & 2+1=3, & 2+2=0, & 2+3=1, \\ 3+0=3, & 3+1=0, & 3+2=1, & 3+3=2. \end{array}$$

Here, $-a$ represents the additive inverse of a, such that $-a + a = 0$ modulo 4. The additive inverses in G are:

$$\begin{array}{ll} -a = 0 & \text{for } a = 0, \\ -a = 3 & \text{for } a = 1, \\ -a = 2 & \text{for } a = 2, \\ -a = 1 & \text{for } a = 3. \end{array}$$

We verify $-a+(a+b)=b$ for all $a, b \in G$ with the left cancellation law:

- Case 1: Let $a = 0, b = 1$:

$$-0+(0+1) = 0+1 = 1 \quad (36)$$

Since $b = 1$, the condition holds.

- Case 2: Let $a = 1, b = 2$:

$$-1+(1+2) = 3+3 \equiv 2 \, mod \, 4 \quad (37)$$

Since $b = 2$, the condition holds.

- Case 3: Let $a = 2, b = 3$:

$$-2+(2+3) = 2+1 \equiv 3 \, mod \, 4 \quad (38)$$

Since, b = 3, the condition holds.

- Case 4: Let a =3,b =0:

$$-3+(3+0) = 1+3 \equiv 0 \, mod \, 4 \quad (39)$$

Since $b = 0$, the condition holds.

For any $a, b \in G$, we compute:

$$-a+(a+b) = (-a+a)+b = 0+b = b \quad (40)$$

Thus, the left cancellation law is satisfied for all $a, b \in G$.

We have demonstrated a numerical example with $G = \{0, 1, 2, 3\}$ under addition modulo 4 that satisfies the left cancellation law:

$$-a+(a+b) = b \quad (41)$$

This verifies the gyrogroup property for $(G, +)$.

Property 10. $\mathrm{gyr}[a, b]x = -(a+b)+\{a+(b+x)\}$. Proof. Let the set $G = \{0, 1, 2, 3\}$ use the addition operation modulo 4. Now, verify the condition:

$$gyr[a,b]x \equiv -(a+b)+\{a+(b+x)\} \, mod \, 4 \quad (42)$$

for all $a, b, x \in G$ as:

a) Let $a = 1$, $b = 2$, and $x = 3$. Now, substitute into the condition:

$$gyr[1,2]3 \equiv -(1+2)+\{1+(2+3)\} \, mod \, 4 \quad (43)$$

b) Compute $-(a+b)$ mod 4 as:

$$-(1+2) = -3 \equiv 1 \, mod \, 4 \quad (44)$$

Thus,

$$-(1+2) \equiv 1 \, mod \, 4 \quad (45)$$

c) Compute $a+(b+x)$ mod 4. First, compute $b+x$ as:

$$b+x = 2+3 = 5 \equiv 1 \, mod \, 4 \quad (46)$$

Next, add a to $(b+x)$ as:

$$a+(b+x) = 1+1 = 2 \equiv 2 \, mod \, 4 \quad (47)$$

Thus:

$$a+(b+x) \equiv 2 \, mod \, 4 \quad (48)$$

d) Combine the results by substituting the values into the condition as:

$$gyr[1,2]3 \equiv -(1+2)+\{1+(2+3)\} \, mod \, 4 \quad (49)$$

Using the computed values:

$$gyr[1,2]3 = 1 + 2 = 3 \equiv 3 \, mod \, 4 \quad (50)$$

Hence, for $a = 1$, $b = 2$, and $x = 3$, the gyrogroup condition holds as gyr[1, 2]3 = 3. It can be concluded that the gyrogroup condition gyr[a, b]$x = -(a + b) + \{a + (b + x)\}$ mod 4 is valid for the set $G = \{0, 1, 2, 3\}$ with the operation addition modulo 4. This satisfies the required condition for G to form a gyrogroup under this operation.

Property 11. gyr[a, b]0 = 0. Proof. Using the proof of Property 10, $x = 0$, we get, gyr[a, b]0 = 0.

Property 12. gyr[a, b]($-x$) = −gyr[a, b]x. Proof. The given condition indicates that the gyrogroup operation should be anti-symmetric with respect to the negative element. Now, we need to verify the given condition for the set $G = \{0, 1, 2, 3\}$ with the addition operation modulo 4:

$$gyr[a,b](-x) \equiv -gyr[a,b]x \, mod \, 4 \quad (51)$$

for all $a, b, x \in G$ as:

a) Let $a = 1$, $b = 2$, and $x = 3$. Now, substitute into the condition:

$$gyr[1,2](-3) \equiv -gyr[1,2](3) \, mod \, 4 \quad (52)$$

b) For the left-hand side, we need to evaluate gyr[a, b]($-x$), which is the gyrogroup operation applied to $-x$. First, calculate the negative of x modulo 4 as:

$$-x \equiv 1 \, mod 4 \quad (53)$$

c) Now, calculate gyr[a, b]($-x$) = gyr[1, 2]1 as:

$$\begin{aligned} \text{gyr}[a,b]x &= -(a + b) + \{a + (b + x)\} \text{ mod } 4 \\ gyr[1,2]1 &\equiv -(1 + 2) + \{1 + (2 + 1)\} \, mod \, 4 \\ &\equiv -3 + \{1 + 3\} \, mod \, 4 \\ &\equiv -3 + 4 \, mod \, 4 \\ &\equiv 1 \, mod \, 4 \end{aligned} \quad (54)$$

d) Now, calculate the right-hand side, which is −gyr[a, b]x. First, calculate gyr[1, 2]3 as:

$$\begin{aligned} gyr[1,2]3 &\equiv -(1 + 2) + 1 + (2 + 3) mod 4 \\ &\equiv -3 + 1 + 5 mod 4 \\ &\equiv -3 + 1 + 5 \, mod \end{aligned} \quad (55)$$

e) Now, calculate the negative of gyr[a, b]x as:

$$-gyr[1,2]3 = -3 \, mod \, 4 \equiv 1 \quad (56)$$

We have obtained:

$$gyr[a,b](-x) = 1 \, and \, - gyr[a,b]x = 1 \quad (57)$$

Thus, the condition gyr[a, b]($-x$) = −gyr[a, b]x holds for the given values of $a = 1$, $b = 2$, and $x = 3$. Hence, the gyrogroup condition gyr[a, b]($-x$) = −gyr[a, b]x is satisfied for the set $G = \{0, 1, 2, 3\}$ with the addition operation modulo 4.

Property 13. gyr[a, 0] = I. Proof. This condition indicates that for any element $a \in G$, applying the gyrogroup operation with 0 should result in the identity element I. For example, in the group $G = \{0, 1, 2, 3\}$ with addition modulo 4, the identity element I is the element that, when added to any element of the group, leaves it unchanged. For a set G with the operation +, the identity element I must satisfy the condition that for every $a \in G$:

$$a + I = a \, for \, all \, a \in G \quad (58)$$

In the case of the set $G = \{0, 1, 2, 3\}$ with addition modulo 4, the identity element I is 0. This is because for any element $a \in G$, we have:

$$a + 0 \equiv a \, mod 4 \quad (59)$$

.

This means that adding 0 to any element *a* in the set *G* returns the same element a, satisfying the condition for the identity element. Thus, in this case, *I* = 0.

Consider the following cases:

- For *a* = 0:

$$gyr[0, 0] = -(0 + 0) + \{0 + (0 + 0)\} \equiv 0 \; mod \; 4 \tag{60}$$

Thus, gyr[0, 0] = 0.

- For *a* = 1:

$$gyr[1, 0] = -(1 + 0) + \{1 + (0 + 0)\} \equiv 0 \; mod \; 4 \tag{61}$$

Thus, gyr[1, 0] = 0.

- For *a* = 2:

$$gyr[2, 0] = -(2 + 0) + \{2 + (0 + 0)\} \equiv 0 \; mod \; 4 \tag{62}$$

Thus, gyr[2, 0] = 0.

- For *a* = 3:

$$gyr[3, 0] = -(3 + 0) + \{3 + (0 + 0)\} \equiv 0 \; mod \; 4 \tag{63}$$

Thus, gyr[3, 0] = 0.

In all cases, adding 0 results in the same element, confirming that *I* = 0. Hence, we have verified that the condition gyr[*a*, 0] = *I* holds for the set *G* = {0, 1, 2, 3} with the addition operation modulo 4. Therefore, the gyrogroup condition is satisfied. Example 2. *(Gyrovector space).* The velocity composition law, specifically in the context of special relativity, can be expressed using the framework of gyrovector spaces. The gyrovector space provides a mathematical foundation for understanding relativistic velocity addition. The relativistic velocity addition (or Thomas motion) is modeled as a gyrogroup operation in a gyrovector space. For velocities **u**, **v** ∈ *G*, where *G* is the set of all velocity vectors with magnitudes less than the speed of light *c*. The velocity composition is given by:

$$u \oplus v = \frac{u + v + \frac{\gamma u}{1+\frac{u.v}{c^2}} u\times(u\times v)}{1+\frac{u.v}{c^2}} \tag{64}$$

Here,

- $\oplus$ represents the gyrogroup operation for velocity addition,
- $\gamma u = \frac{1}{\sqrt{1-\frac{|u|^2}{c^2}}}$ is the Lorentz factor for velocity u,
- u · v is the dot product of u and v,
- u × v is the cross product of u and v.

The following is an example of velocity addition computation in one dimension using the gyrovector space framework.

- Let u = 0.5*c* and **v** = 0.4*c*, where *c* is the speed of light.
- The relativistic velocity addition formula in 1D simplifies to:

$$u \oplus v = \frac{u+v}{1+\frac{uv}{c^2}} \tag{65}$$

Final calculation is provided below:

1. Substitute *u* = 0.5*c* and *v* = 0.4*c*:

$$u \oplus v = \frac{0.5c+0.4c}{1+\frac{(0.5c)(0.4c)}{c^2}} \tag{66}$$

2. Simplify the numerator:

$$u \oplus v = \frac{0.9c}{1+\frac{0.2c^2}{c^2}}$$

(67)

3. Simplify the denominator:

$$u \oplus v = \frac{0.9c}{1+0.2}$$

(68)

4. Perform the division:

$$u \oplus v = \frac{0.9c}{1.2} = 0.75c \tag{69}$$

The resulting velocity after relativistic addition is: 0.75c.

4. AMBIGUOS SET

This section covers the fundamental definition of an ambiguous set, and ambiguous space, as well as a number of related definitions, formulae, and properties [20]-[28].

Definition 2 (Ambiguous set). [29]-[35]. Let $X = \{x\}$ denote the fixed universe of discourse for any event x. An ambiguous set C for $x \in X$ is defined as:

$$C = \{x, \theta C(x), \varphi C(x), \sigma C(x), \omega C(x) \mid x \in X\} \tag{70}$$

where, the membership degrees $\theta_C(x)$, $\varphi_C(x)$, $\sigma_C(x)$, $\omega_C(x)$: $X \longrightarrow [0, 1]$ with the condition:

$$0 \leq \theta C(x) + \varphi C(x) + \sigma C(x) + \omega C(x) \leq 2, for\ any\ event\ x \tag{71}$$

Here, θ_C, φ_C, σ_C, and ω_C represent four distinct membership functions termed as true, false, partially true, and partially false, respectively. These membership functions are collectively referred to as ambiguous. In Equation (70), the values $\theta_C(x)$, $\varphi_C(x)$, $\sigma_C(x)$, $\omega_C(x)$, which lie within [0, 1], are known as the true, false, partially true, and partially false membership degrees of $x \in X$ in C, respectively.

An ambiguous set C can also be defined for discrete and continuous cases as:

- Discrete Case: For a discrete universe $X = \{x_1, x_2, \ldots, x_n\}$, the ambiguous set C is defined as:

$$C = \{xi, \theta C(xi), \varphi C(xi), \sigma C(xi), \omega C(xi) \mid xi \in X, i = 1,2, \ldots, n\} \tag{72}$$

where, the membership degrees $\theta_C(x_i)$, $\varphi_C(x_i)$, $\sigma_C(x_i)$, and $\omega_C(x_i)$ satisfy:

$$0 \leq \theta C(xi) + \varphi C(xi) + \sigma C(xi) + \omega C(xi) \leq 2, for\ each\ xi \in X. \tag{73}$$

- Continuous Case: For a continuous universe X, the ambiguous set C is represented as:

$$C = \{x, \theta C(x), \varphi C(x), \sigma C(x), \omega C(x) \mid x \in X\} \tag{74}$$

where, the membership degrees $\theta_C(x)$, $\varphi_C(x)$, $\sigma_C(x)$, and $\omega_C(x)$: $X \longrightarrow [0, 1]$ satisfy:

$$0 \leq \theta C(x) + \varphi C(x) + \sigma C(x) + \omega C(x) \leq 2, for\ any\ x \in X. \tag{75}$$

Next, we provide the concept of a *single-valued ambiguous number (SVAN)*.

Definition 3 (SVAN). To represent an event $x \in X$ in an ambiguous set C, we can use the notation $x = \langle \theta_x, \varphi_x, \sigma_x, \omega_x \rangle$, which we call an SVAN. Here, θ_x, φ_x, σ_x, ω_x correspond to the membership degrees $\theta_C(x)$, $\varphi_C(x)$, $\sigma_C(x)$, $\omega_C(x)$ of the event x in the ambiguous set C, respectively. These membership degrees must satisfy the condition:

$$0 \leq \theta x + \varphi x + \sigma x + \omega x \leq 2, \forall x \in X. \tag{76}$$

This ensures that the total membership of each event x in the set C remains within the bounds of ambiguity.

5. AMBIGUOS SPACE

This section introduces the concept of ambiguous space.

Definition 4 (Ambiguous space). Let $(C, \oplus, \otimes, \lambda \cdot x, x^{\lambda})$ be an ambiguous groupoid, where C is a set of SVANs, and $\oplus$, $\otimes$, $\lambda \cdot x$, x^{λ} represent the operations for addition, multiplication, scalar (λ) multiplication for x, exponentiation for x (i.e., x is raised to the power of λ), respectively, on these SVANs. This structure forms an *ambiguous space*, and (C, ⊕, ⊗, $\lambda \cdot x$, x^λ) is called an *ambigroup* if the following operational laws, based on the Lemmas, are satisfied for any two SVANs x and y in C.

Operational Laws and Lemmas: For the two SVANs $x = \langle\theta_x, \varphi_x, \sigma_x, \omega_x\rangle$ and $y = \langle\theta_y, \varphi_y, \sigma_y, \omega_y\rangle$, the following operational laws hold in an ambiguous space:

- Lemma 1 (Addition):

$$x \oplus y = \theta x + \theta y - \theta x \cdot \theta y, \varphi x \cdot \varphi y, \sigma x \cdot \sigma y, \omega x \cdot \omega y . \tag{77}$$

This lemma defines the addition operation ⊕ between two SVANs.

- Lemma 2 (Multiplication):

$$x \otimes y = \theta x \cdot \theta y, \varphi x \cdot \varphi y, \sigma x \cdot \sigma y, \omega x \cdot \omega y . \tag{78}$$

This lemma defines the multiplication operation ⊗ between two SVANs.

- Lemma 3a (Scalar multiplication for x):

$$\lambda \cdot x = 1 - (1 - \theta x)\lambda, 1 - (1 - \varphi x)\lambda, 1 - (1 - \sigma x)\lambda, 1 - (1 - \omega x)\lambda . \tag{79}$$

Here, scalar λ belongs to (0, 1).

- Lemma 3b (Scalar multiplication for y):

$$\lambda \cdot y = 1 - (1 - \theta y)\lambda, 1 - (1 - \varphi y)\lambda, 1 - (1 - \sigma y)\lambda, 1 - (1 - \omega y)\lambda . \tag{80}$$

- Lemma 4a (Exponentiation for x):

$$x\lambda = (\theta x)\lambda, (\varphi x)\lambda, (\sigma x)\lambda, (\omega x)\lambda , \lambda \in (0,1). \tag{81}$$

Here, SVAN x is raised to the power of λ.

- Lemma 4b (Exponentiation for y):

$$y\lambda = (\theta y)\lambda, (\varphi y)\lambda, (\sigma y)\lambda, (\omega y)\lambda , \lambda \in (0,1). \tag{82}$$

Similar to Lemma 4a, this lemma defines the exponentiation operation for the SVAN y.

The following theorems explore operational laws in the context of ambiguous space.

Theorem 1. Let $r = x \oplus y$, then r is also an SVAN.

Proof: Since, $x = \langle\theta_x, \varphi_x, \sigma_x, \omega_x\rangle$ and $q = \langle\theta_y, \varphi_y, \sigma_y, \omega_y\rangle$ are two SVANs, and $\theta_x, \theta_y \in [0, 1]$, $\varphi_x, \varphi_y \in [0, 1]$, $\sigma_x, \sigma_y \in [0, 1]$, $\omega_x, \omega_y \in [0, 1]$, then by following Lemma 1, we have:

$0 \le \theta_x + \theta_y - \theta_x \cdot \theta_y \le 1,$

$0 \le \varphi_x \cdot \varphi_y \le 1,$

$0 \le \sigma_x \cdot \sigma_y \le 1,$ and

$0 \le \omega_x \cdot \omega_y \le 1/2.$

Since, $0 \le \theta_x + \varphi_x + \sigma_x + \omega_x \le 2$ and $0 \le \theta_y + \varphi_y + \sigma_y + \omega_y \le 2$. Therefore, $0 \le (\theta_x + \theta_y - \theta_x \cdot \theta_y) + (\varphi_x \cdot \varphi_y) + (\sigma_x \cdot \sigma_y) + (\omega_x \cdot \omega_y) \le 2$. Thus, $r = \langle\theta_r, \varphi_r, \sigma_r, \omega_r\rangle$ is a SVAN by satisfying the condition $0 \le \theta_r + \varphi_r + \sigma_r + \omega_r \le 2$. □

Theorem 2. Let $r = x \otimes y$, then r is also an SVAN.

Proof: Since, $x = \langle\theta_x, \varphi_x, \sigma_x, \omega_x\rangle$ and $y = \langle\theta_y, \varphi_y, \sigma_y, \omega_y\rangle$ are two SVANs, and $\theta_x, \theta_y \in [0, 1]$, $\varphi_x, \varphi_y \in [0, 1]$, $\sigma_x, \sigma_y \in [0, 1]$, $\omega_x, \omega_y \in [0, 1]$, then by following Lemma 2, we have:

$0 \le (\theta_x \cdot \theta_y) \le 1$

$0 \le (\varphi_x \cdot \varphi_y) \le 1,$

$0 \le (\sigma_x \cdot \sigma_y) \le 1,$ and

$0 \le (\omega_x \cdot \omega_y) \le 1.$

Since, $0 \le \theta_x + \varphi_x + \sigma_x + \omega_x \le 2$ and $0 \le \theta_y + \varphi_y + \sigma_y + \omega_y \le 2$. Therefore, $0 \le (\theta_x \cdot \theta_y) + (\varphi_x \cdot \varphi_y) + (\sigma_x \cdot \sigma_y) + (\omega_x \cdot \omega_y) \le 2$. Thus, $r = \langle \theta_r, \varphi_r, \sigma_r, \omega_r \rangle$ is a SVAN by satisfying the condition $0 \le \theta_r + \varphi_r + \sigma_r + \omega_r \le 2$, where $\theta_r = (\theta_x \cdot \theta_y)$, $\varphi_r = (\varphi_x \cdot \varphi_y)$, $\sigma_r = (\sigma_x \cdot \sigma_y)$, and $\omega_r = (\omega_x \cdot \omega_y)$. □

Theorem 3. Let $r = \lambda \cdot x$, then r is also an SVAN.

Proof: Since, $x = \langle \theta_x, \varphi_x, \sigma_x, \omega_x \rangle$ is a SVAN, and $\theta_x \in [0, 1]$, $\varphi_x \in [0, 1]$, $\sigma_x \in [0, 1]$, $\omega_x \in [0, 1]$, then by following

Lemma 3a, we have:

$0 \le 1 - (1 - \theta_x)^\lambda \le 1$,

$0 \le 1 - (1 - \varphi_x)^\lambda \le 1$,

$0 \le 1 - (1 - \sigma_x)^\lambda \le 1$, and

$0 \le 1 - (1 - \omega_x)^\lambda \le 1$.

Since, $0 \le \theta_x + \varphi_x + \sigma_x + \omega_x \le 2$. Therefore, $0 \le \{1 - (1-\theta_x)^\lambda\} + \{1 - (1-\varphi_x)^\lambda\} + \{1 - (1-\sigma_x)^\lambda\} + \{1 - (1-\omega_x)^\lambda\} \le 2$. Thus, $r = \langle \theta_r, \varphi_r, \sigma_r, \omega_r \rangle$ is an SVAN by satisfying the condition $0 \le \theta_r + \varphi_r + \sigma_r + \omega_r \le 2$, where $\theta_r = 1 - (1 - \theta_r)^\lambda$, $\varphi_r = 1 - (1 - \varphi_r)^\lambda$, $\sigma_r = 1 - (1 - \sigma_r)^\lambda$, and $\omega_r = 1 - (1 - \omega_r)^\lambda$.

Theorem 4. Let $r = \lambda \cdot y$; then r is also an SVAN

Proof: Same as proof of Theorem 3.

Theorem 5. Let $r = x^\lambda$; then r is also an SVAN.

Proof: Since, $x = \langle \theta_x, \varphi_x, \sigma_x, \omega_x \rangle$ is a SVAN, and $\theta_x \in [0, 1]$, $\varphi_x \in [0, 1]$, $\sigma_x \in [0, 1]$, $\omega_x \in [0, 1]$, then by following

Lemma 4a, we have:

$0 \le (\theta_x)^\lambda \le 1$,

$0 \le (\varphi_x)^\lambda \le 1$,

$0 \le (\sigma_x)^\lambda \le 1$, and

$0 \le (\omega_x)^\lambda \le 1$.

Thus, $r = \langle \theta_r, \varphi_r, \sigma_r, \omega_r \rangle$ is a SVAN by satisfying the condition $0 \le \theta_r + \varphi_r + \sigma_r + \omega_r \le 2$, where $\theta_r = (\theta_x)^\lambda$, $\varphi_r = (\varphi_x)^\lambda$,

$\sigma_r = (\sigma_x)^\lambda$, and $\omega_r = (\omega_x)^\lambda$.

Theorem 6. Let $r = y^\lambda$, then r is also an SVAN.

Proof: Same as proof of Theorem 5.

Theorem 7. For two SVANs x and y, it follows:

(1) $x \otimes y = y \otimes x$;

(2) $(x \otimes y)^\lambda = (y \otimes x)^\lambda$;

(3) $x^\lambda 1 \otimes x^\lambda 2 = (x)^\lambda 1+\lambda 2$.

Proof (1): By following Lemma 2, we have:

$$
\begin{aligned}
x \otimes y &= \langle (\theta_x \cdot \theta_y), (\varphi_x \cdot \varphi_y), (\sigma_x \cdot \sigma_y), (\omega_x \cdot \omega_y) \rangle \\
&= \langle (\theta_y \cdot \theta_x), (\varphi_y \cdot \varphi_x), (\sigma_y \cdot \sigma_x), (\omega_y \cdot \omega_x) \rangle \\
&= y \otimes x. \ \square
\end{aligned}
$$

Proof (2): Since, $x \otimes y = \langle (\theta_x \cdot \theta_y), (\varphi_x \cdot \varphi_y), (\sigma_x \cdot \sigma_y), (\omega_x \cdot \omega_y) \rangle$. Then, by Lemma 2, it follows:

$$
\begin{aligned}
(x \otimes y)^\lambda &= \langle (\theta_x \cdot \theta_y)^\lambda, (\varphi_x \cdot \varphi_y)^\lambda, (\sigma_x \cdot \sigma_y)^\lambda, (\omega_x \cdot \omega_y)^\lambda \rangle \\
&= \langle (\theta_y \cdot \theta_x)^\lambda, (\varphi_y \cdot \varphi_x)^\lambda, (\sigma_y \cdot \sigma_x)^\lambda, (\omega_y \cdot \omega_x)^\lambda \rangle
\end{aligned}
$$

.

$= (y \otimes x)^{\lambda}$. □

Proof (3): By following Lemma 4a, we have $x^{\lambda}1 = \langle (\theta_x)^{\lambda}1, (\varphi_x)^{\lambda}1, (\sigma_x)^{\lambda}1, (\omega_x)^{\lambda}1 \rangle$. Similarly, by following Lemma 4b, we have $x^{\lambda}2 = \langle (\theta_x)^{\lambda}2, (\varphi_x)^{\lambda}2, (\sigma_x)^{\lambda}2, (\omega_x)^{\lambda}2 \rangle$.

$$x^{\lambda}1 \otimes x^{\lambda}2 = \langle (\theta_x)^{\lambda}1 \otimes (\theta_x)^{\lambda}2, (\varphi_x)^{\lambda}1 \otimes (\varphi_x)^{\lambda}2, (\sigma_x)^{\lambda}1 \otimes (\sigma_x)^{\lambda}2, (\omega_x)^{\lambda}1 \otimes (\omega_x)^{\lambda}2 \rangle$$

$$= \langle (\theta x)\lambda 1{+}\lambda 2, (\varphi x)\lambda 1{+}\lambda 2, (\sigma x)\lambda 1{+}\lambda 2, (\omega x)\lambda 1{+}\lambda 2 \rangle$$

$$= (x)^{\lambda}1{+}\lambda 2 .$$

6. APPLICATION OF AMBIGUOUS SPACE FOR CUSTOMER SEGMENTATION

Consider the classification of customers into ambiguous categories:

- Highly Satisfied (HS)
- Neutral (N)
- Dissatisfied (D)
- Step 1: Represent customer feedback using SVANs. Let each customer's feedback be evaluated based on four membership degrees:

– θ_x: Membership degree of the customer being Highly Satisfied.
– φ_x: Membership degree of the customer being Dissatisfied.
– σ_x: Partial truth of being Neutral.
– ω_x: Partial falsity of being Neutral.

For example, feedback from customer C_1 is represented as:

$$xC1 = \langle \theta xC1, \varphi xC1, \sigma xC1, \omega xC\rangle 1 = \langle 0.8,0.1,0.6,0.2\rangle \quad (83)$$

This means:

– $\theta_{xc_1} = 0.18$: 18% Highly Satisfied.
– $\varphi_{xc_1} = 0.74$: 74% Dissatisfied.
– $\sigma_{xc_1} = 0.17$: 17% Neutral (partial truth).
– $\omega_{xc_1} = 0.68$: 68% not Neutral (partial falsity).

Another customer, C_2's feedback:

$$yC2 = \langle \theta yC2, \varphi yC2, \sigma yC2, \omega yC2 \rangle = \langle 0.5,0.3,0.7,0.4\rangle \quad (84)$$

This means:

– $\theta_{yc_2} = 0.27$: 27% Highly Satisfied.
– $\varphi_{yc_2} = 0.65$: 65% Dissatisfied.
– $\sigma_{yc_2} = 0.25$: 25% Neutral (partial truth).
– $\omega_{yc_2} = 0.59$: 59% not Neutral (partial falsity).

Step 2: Perform an ambiguous operation. Using addition in an ambiguous space (Lemma 1):

$$xC1 \oplus yC2 = \langle \theta xC1 + \theta yC2 - \theta xC1 \cdot \theta yC2, \varphi xC1 \cdot \varphi yC2, \sigma xC1 \cdot \sigma yC2, \omega xC1 \cdot \omega yC2\rangle$$

$$xC1 \oplus yC2 = \langle 0.8 + 0.5 - (0.8 \cdot 0.5), 0.74 \cdot 0.65, 0.17 \cdot 0.25, 0.68 \cdot 0.59\rangle$$

$$xC1 \oplus yC2 = \langle 0.4014, 0.481, 0.0425, 0.4012\rangle \quad (85)$$

The result of the calculation $x_C1 \oplus y_C2 = \langle 0.4014, 0.481, 0.0425, 0.4012\rangle$ can be interpreted as follows:

b. Highly Satisfied (θ): The combined membership degree for being "Highly Satisfied" is 0.4014, or approximately 40.14%. This indicates a moderate level of satisfaction after combining the assessments of x_C1 and y_C2 .
c. Dissatisfied (φ): The combined membership degree for being "Dissatisfied" is 0.481, or approximately 48.1%. This suggests that dissatisfaction remains a prominent sentiment after the combination.
d. Neutral (σ): The combined membership degree for being "Neutral" (partial truth) is 0.0425, or approximately 4.25%. This is significantly lower than the individual components, indicating a very low neutrality after combining the assessments.
e. Not Neutral (ω): The combined membership degree for being "Not Neutral" (partial falsity) is 0.4012, or approximately 40.12%. This reflects a moderate degree of falsity in being neutral after the combination.

It indicates that dissatisfaction (φ) remains the dominant sentiment at 48.1%, followed by moderate levels of high satisfaction (θ = 40.14%) and "Not Neutral" (ω = 40.12%). Neutrality (σ = 4.25%) is minimal, suggesting that most responses lean clearly towards either satisfaction or dissatisfaction.

7. CONCLUSIONS AND FUTURE DIRECTIONS

This study established a comparative framework that linked hyperbolic space, gyrovector space, and ambiguous space, thereby expanding traditional mathematical structures to accommodate domains requiring more nuanced representations of uncertainty. Hyperbolic space, widely used in computational geometry and network analysis, offered a non-Euclidean foundation, while gyrovector space improved algebraic efficiency in hyperbolic operations. The introduction of ambiguous sets, which feature four distinct membership degrees, viz., true, false, partially true, and partially false, enabled a richer representation of uncertainty. This study also introduced the SVAN, a notation for representing events within an ambiguous set. The ambiguous space is then defined as a structure consisting of SVANs, with operations for addition, multiplication, scalar multiplication, and exponentiation. This space forms an ambigroup, satisfying operational laws for these operations, such as addition and multiplication between two SVANs and scalar multiplication for individual elements. These elements integrate seamlessly into the broader framework, contributing to the emergence of ambiguous space as a novel mathematical structure with diverse potential applications.

A key application explored in this study was customer segmentation, where ambiguous space enabled a more flexible classification framework. Unlike conventional clustering techniques that relied on strict boundaries, ambiguous space accommodated partial truths and uncertainties, making it well-suited for real-world decision-making. This approach could be further extended to various fields such as sentiment analysis, medical diagnosis, and financial risk assessment [36][37].

REFERENCES

[1] J. Gray, "Euclidean And Non-Euclidean Geometry," *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, vol. 2, p. 877, 2004.

[2] J. W. Milnor, "Hyperbolic Geometry: The First 150 Years," *Bulletin of the American Mathematical Society*, vol. 6, no. 1, pp. 9–24, 1982. https://doi.org/10.1090/S0273-0979-1982-14958-8.

[3] J. Stillwell, " Sources Of Hyperbolic Geometry," *American Mathematical Society,* 1996, no. 10. https://doi.org/10.1090/hmath/010.

[4] B. Iversen, "Hyperbolic Geometry*,"* *Cambridge University Press,* 1992, no. 25. https://doi.org/10.1017/CBO9780511569333.

[5] W. F. Reynolds, "Hyperbolic Geometry on a Hyperboloid," *The American Mathematical Monthly*, vol. 100, no. 5, pp. 442–455, 1993. https://doi.org/10.1080/00029890.1993.11990430.

[6] D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, and M. Bogunа́, "Hyperbolic Geometry Of Complex Net- Works," *Physical Review E*, vol. 82, no. 3, p. 036106, 2010. https://doi.org/10.1103/PhysRevE.82.036106.

[7] J. W. Cannon, W. J. Floyd, R. Kenyon, and W. R. Parry, "Hyperbolic Geometry," *Flavors of geometry*, vol. 31, pp. 59–110, 1997.

[8] J. M. Lee, "Smooth Maps. Springer," *New York,* 2012, pp. 32–49. https://doi.org/10.1007/978-1-4419-9982-5_2.

[9] A. A. Ungar, "Analytic Hyperbolic Geometry: Mathematical Foundations And Applications," World Scientific, 2005. https://doi.org/10.1142/5914.

[10] F. Papadopoulos, M. Kitsak, M. Á . Serrano, M. Boguna, and D. Krioukov, "Popularity Versus Similarity In Growing Networks," *Nature, vol. 489, no. 7417, pp. 537–540,* 2012. https://doi.org/10.1038/nature11459.

[11] I. Athal et al., "Combined Impact of Radiation and Chemical Reaction on MHD Hyperbolic Tangent Nanofluid Boundary Layer Flow Past a Stretching Sheet," *Modern Physics Letters B*, vol. 38, no. 16, p. 2341010, 2024. https://doi.org/10.1142/S0217984923410105.

[12] N. Khan, M. Zeeshan, M. Hashmi, and M. Inc, "Variable Thermal Conductivity And Chemical Reaction Aspects In MHD Tangent Hyperbolic Nanofluid Flow Over An Exponentially Stretching Surface," *Modern Physics Letters B*, vol. 38, no. 11, p. 2450066, 2024. https://doi.org/10.1142/S0217984924500660.

[13] H. Wu et al., "Tunable Biaxial Hyperbolic Dispersion and Negative Refraction In Graphite," *Modern Physics Letters B*, vol. 34, no. 11, p. 2050110, 2020. https://doi.org/10.1142/S0217984920501109.

[14] Y.-M. Chu *et al.*, "Joule Heating, Activation Energy And Modified Diffusion Analysis For 3D Slip Flow Of Tangent Hyperbolic Nanofluid with Gyrotactic Microorganisms," *Modern Physics Letters B*, vol. 35, no. 16, p. 2150278, 2021. https://doi.org/10.1142/S021798492150278X.

[15] L. Mari, J. R. de Oliveira, A. Savas-Halilaj, and R. S. de Sena, "Conformal Solitons For The Mean Curvature Flow In Hyperbolic Space," *Annals of Global Analysis and Geometry*, vol. 65, no. 2, p. 19, 2024. https://doi.org/10.1007/s10455-024-09947-y.

[16] A. Khaldi, M. Maouni, and A. Ouaoua, "Asymptotic Behavior of Solution for a Fractional P-Kirchhoff Type Hyperbolic Equation with Variable Exponent," *International Journal of Applied and Computational Mathematics*, vol. 10, no. 2, p. 51, 2024. https://doi.org/10.1007/s40819-023-01657-6.

[17] G. Mandal, S. Das, S. Dutta, L. N. Guin, and S. Chakravarty, "A Comparative Study Of Allee Effects and Fear- Induced Responses: Exploring Hyperbolic and Ratio-Dependent Models," *International Journal of Applied and Computational Mathematics*, vol. 10, no. 5, p. 147, 2024. https://doi.org/10.1007/s40819-024-01773-x.

[18] A. A. Mahmud, T. Tanriverdi, K. A. Muhamad, and H. M. Baskonus, “An Investigation Of The Influence Of Time Evolution on The Solution Structure Using Hyperbolic Trigonometric Function Methods,” *International Journal of Applied and Computational Mathematics*, vol. 10, no. 4, p. 137, 2024. https://doi.org/10.1007/s40819-024-01769-7.
[19] A. A. Ungar, “Gyrovector Spaces and Their Differential Geometry,” *Nonlinear Funct. Anal. Appl.*, vol. 10, no. 5, pp. 791–834, 2005.
[20] P. Singh, “From Ambiguous Sets to Single-Valued Ambiguous Complex Numbers: Applications In Mandelbrot Set Generation and Vector Directions,” *Modern Physics Letters B*, vol. 39, no. 14, p. 2450509, 2025. https://doi.org/10.1142/S0217984924505092.
[21] P. Singh, “A Journey into Ambiguous Set Theory: Exploring Ambiguity,” Cambridge *Scholars Publishing*, 2025.
[22] P. Singh and S. S. Bose, “Ambiguous D-Means Fusion Clustering Algorithm Based On The Ambiguous Set Theory: Special Application in Clustering of CT Scan Images Of COVID-19,” *Knowledge-Based Systems*, vol. 231, p. 107432, 2021. https://doi.org/10.1016/j.knosys.2021.107432.
[23] P. Singh and Y.-P. Huang, “Membership Functions, Set-Theoretic Operations, Distance Measurement Methods Based on Ambiguous Set Theory: A Solution to a Decision-Making Problem In Selecting The Appropriate Colleges,” *Int. J. Fuzzy Syst.*, vol. 25, pp. 1311–1326, 2023. https://doi.org/10.1007/s40815-023-01468-3.
[24] P. Singh, “Data-Driven Ambiguous Cognitive Map for Complex Decision-Making In Supply Chain Management,” *Journal of Computational Mathematics and Data Science*, vol. 14, p. 100110, 2025. https://doi.org/10.1016/j.jcmds.2025.100110.
[25] P. Singh, “Ambiguous Sets and Various Distance Measurements,” *Soft Computing and Its Engineering Appli- cations*, ser. Communications in Computer and Information Science, vol. 2430, 2025.
[26] P. Singh, “A Novel Model to Deal with Ambiguous and Complex Time Series: Application to Sunspots Forecasting,” *Knowledge-Based Systems*, vol. 329, p. 114257, 2025. https://doi.org/10.1016/j.knosys.2025.114257.
[27] P. Singh, B. Saini, and Y.-P. Huang, “AECA: An Ambiguous-Entropy Clustering Algorithm For The Analysis of Resting-State Fmriss of Human Brain and Their Functional Connections,” *Modern Physics Letters B*, vol. 39, no. 18, p. 2550023, 2025. https://doi.org/10.1142/S021798492550023X.
[28] P. Singh and T. Liao, “Multi-Criteria Group Decision-Making Using Ambiguous Sets, Weibull Distribution, And Aggregation Operators: A Case Study in Optimal Vendor Selection For Office Supplies,” *Systems and Soft Computing*, vol. 7, p. 200283, 2025. https://doi.org/10.1016/j.sasc.2025.200283.
[29] P. Singh, Y.-P. Huang, and T.-T. Lee, “A Novel Ambiguous Set Theory To Represent Uncertainty and Its Application to Brain MR Image Segmentation,” *Proc. of IEEE Int. Conf. on Systems, Man and Cybernetics (SMC)*, Bari, Italy, 2019, pp. 2460–2465. https://doi.org/10.1109/SMC.2019.8914080.
[30] P. Singh, “An Investigation of Ambiguous Sets and Their Application to Decision-Making from Partial Order to Lattice Ambiguous Sets,” *Decision Analytics Journal*, vol. 8, p. 100286, 2023. https://doi.org/10.1016/j.dajour.2023.100286.
[31] P. Singh, “A General Model of Ambiguous Sets to A Single-Valued Ambiguous Numbers with Aggregation Operators,” *Decision Analytics Journal*, vol. 8, p. 100260, 2023. https://doi.org/10.1016/j.dajour.2023.100260.
[32] P. Singh, “Ambiguous set theory: A New Approach to Deal with Unconsciousness and Ambiguousness Of Human Perception,” *Journal of Neutrosophic and Fuzzy Systems*, vol. 5, no. 1, pp. 52–58, 2023. https://doi.org/10.54216/JNFS.050106.
[33] P. Singh and Y.-P. Huang, “An Ambiguous Edge Detection Method for Computed Tomography Scans Of Coronavirus Disease 2019 Cases,” *IEEE Trans. on Systems, Man, and Cybernetics: Systems*, vol. 54, no. 1, pp. 352–364, 2024. https://doi.org/10.1109/TSMC.2023.3307393.
[34] P. Singh and Y.-P. Huang, “AKDC: Ambiguous Kernel Distance Clustering Algorithm For COVID-19 CT Scans Analysis,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 54, no. 10, pp. 6218–6229, 2024. https://doi.org/10.1109/TSMC.2024.3418411.
[35] P. Singh and Y.-P. Huang, “A four-valued ambiguous logic: application in designing ambiguous inference system for control systems,” *International Journal of Fuzzy Systems*, vol. 25, no. 8, pp. 2921–2938, 2023. https://doi.org/10.1007/s40815-023-01582-2.
[36] A. Fitriani, A. Amirullah, and K. Bhumkittipich, “Power quality enhancement using single phase shunt active filter based ANFIS supplied by photovoltaic,” *Vokasi Unesa Bull. Eng. Technol. Appl. Sci.*, vol. 2, no. 3, pp. 558–580, 2025. https://doi.org/10.26740/vubeta.v2i3.39071.
[37] J. A. Obari, S. T. A, I. M. A, and A. B. H, “An emission and weight of vehicles-based road traffic congestion pricing system and control with consideration of investment worthiness,” *Vokasi Unesa Bull. Eng. Technol. Appl. Sci.*, vol. 2, no. 3, pp. 401–411, 2025. https://doi.org/10.26740/vubeta.v2i3.38528.