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Soliton Dynamics in Combined KdV-mKdV and KdV-nKdV Models: A Riccati-Bernoulli Sub ODE Approach

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ABSTRACT

We investigate new soliton solutions of two coupled nonlinear systems: the combined KdV–nKdV and the KdV–mKdV equations. Using the Riccati–Bernoulli Sub-ODE (RBSODE) method with r = 0, the models are reduced to tractable algebraic forms that yield explicit trigonometric, hyperbolic, and exponential-type soliton families. The analytical procedure reveals parameter conditions under which compressive and oscillatory solitons emerge, such as $\frac{\delta}{\alpha^2(\alpha+\delta)} > 0$ for localized bright solitons. A systematic parameter study quantifies how amplitude, width, and velocity vary with the nonlinear coefficients α, δ, p, q. Comparison with existing results (Wazwaz 2017) shows that our solutions recover known families as special cases while extending them to additional parameter regimes. Physical implications are discussed in the context of nonlinear wave propagation in dispersive media, where the balance between quadratic and cubic nonlinearities governs soliton shape and robustness. The results demonstrate that the RBSODE approach provides a flexible symbolic framework for constructing diverse soliton families and analyzing their parameter dependence in coupled nonlinear systems.

1. INTRODUCTION

The study of nonlinear partial differential equations (PDEs) has profoundly advanced our understanding of complex wave phenomena across diverse scientific domains, particularly in fluid dynamics and optical physics [1]-[3]. At the forefront of these investigations stands the Korteweg-de Vries (KdV) equation, renowned for its ability to model soliton behavior, namely localized wave packets that maintain their structural integrity through a delicate balance between nonlinear steepening and dispersive spreading [4]-[6]. However, contemporary physical systems frequently exhibit additional nonlinear features that transcend the descriptive capabilities of the classical KdV framework. This limitation has spurred the development of enhanced formulations, particularly the coupled KdV-modified KdV (mKdV) and KdV-negative-order KdV (nKdV) systems, which incorporate higher- order nonlinear effects critical for modern applications [7]-[13].

The KdV-mKdV system represents a significant advancement over its classical counterpart by incorporating cubic nonlinearities [14]-[21]. This extension is particularly relevant for modeling pulse propagation in one-dimensional dispersive media, where the interplay between quadratic and cubic nonlinear terms gives rise to novel soliton behaviors [22]-[29]. Recent studies have demonstrated that such combined nonlinearities can produce soliton solutions with unique spectral characteristics and interaction properties, opening new possibilities for optical signal processing. The modified KdV component introduces an additional dimension to wave dynamics analysis, enabling more precise modeling of amplitude-dependent phenomena in photonic crystals and nonlinear optical fibers [30]-[37].

Conversely, the KdV-nKdV system exhibits fundamentally different behavior due to its inclusion of negative nonlinear terms [9]. These terms induce remarkable compressive effects that invert conventional soliton dynamics, challenging established paradigms of wave propagation. Physical manifestations of these

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compressive nonlinearities have been observed in certain plasma media and nonlinear transmission lines, enabling unprecedented control over pulse shaping and wave localization. The nKdV component's ability to counteract natural dispersion makes it particularly valuable for applications requiring pulse compression or energy localization in optical communication systems [38] [39].

The combined KdV-nKdV equation, expressed as [1]:

$$u_{xt} + 6u_x u_{xx} + u_{xxx} + u_{xxt} + 4u_x u_{xt} + 2u_{xx} u_t = 0 \tag{1}$$

Embodies these analytical challenges through its intricate coupling of dispersive and nonlinear terms. This fourth-order nonlinear system arises from the synthesis of conventional KdV dynamics with negative-order recursion operators, yielding complex wave interactions that elude simple perturbation analysis [1]. The additional terms uxxt and uxt introduce higher-order temporal dispersion effects that significantly alter solution characteristics compared to standard KdV behavior [40]-[44]. Wazwaz combined the negative KdV recursion operator and the Korteweg-de Vries (KdV) recursion operator to introduce the fourth-order nonlinear equation, as reported in [1]. Various methods have been employed to seek solutions for the combined KdV-nKdV equation. These methods include the Tan-Cot method[1], modified Hirota's method [45]-[47], Lie symmetry approach [48]-[50], among others. While [1] used Tan-Cot for KdV-nKdV, solutions were limited to specific parameter regimes. The Lie symmetry approach [39] lacks generality for higher-order nonlinearities.

In parallel, the coupled KdV-mKdV system [38]:

$$u_t + puu_x + qu^2 u_x + ru_{xxx} = 0 \tag{2}$$

Where p, q, and r represent physical parameters, presents its own set of analytical difficulties. The coexistence of quadratic (uu_x) and cubic ($u^2 u_x$) nonlinear terms creates a rich solution space that includes both bright and dark soliton varieties, along with more exotic wave forms [38].

Numerous methods, including the improved trigonometric function method [51], the auxiliary difference equation method [52], the Jacobi elliptic function method [53], and many more, have been employed to search for the solution of the combined KdV–mKdV.

The RBSODE approach reduces the original nonlinear PDEs into low-order algebraic systems, which can be solved symbolically with relatively modest computational effort. Compared to methods such as Hirota's bilinear method, which require constructing perturbative expansions, or Lie symmetry analysis, which involves solving invariant conditions, the RBSODE method offers a direct symbolic route to soliton solutions. While it does not guarantee completeness or extend naturally to multi-soliton cases, it is computationally efficient for deriving explicit single-soliton families, making it suitable for analyzing higher-order nonlinearities in coupled KdV systems. During the solution process, some parameter combinations yield algebraic systems with no consistent nontrivial solution or lead to expressions equivalent to previously obtained results. In such cases, these redundant outcomes are excluded to avoid repetition. Only parameter sets that produce distinct functional forms or that reveal new parameter regimes are presented in the results section. This study aims to achieve three primary objectives:

- Derivation of Novel Soliton solutions for KdV-nKdV and KdV-mKdV systems based on their distinct non- linear signatures using Ricatti-Bernoulli sub-ODE (RBSODE) method.
- Quantitative Analysis of Soliton Dynamics. We aim to establish a quantitative relationship between system parameters (p, q, r) and soliton properties, including amplitude, velocity, and stability thresholds, using 3 Dimensional figures.
- We will conduct a rigorous comparison of the Riccati-Bernoulli sub-ODE (RBSODE) approach with established methods, such as the Hirota bilinear method, to evaluate their solution completeness and computational efficiency.

The significance of this work extends beyond theoretical considerations. In optical communications, the derived solutions provide concrete design principles for managing nonlinear pulse distortion in next-generation fiber systems. For photonic device engineering, the results offer guidelines for controlling soliton interactions in integrated waveguide arrays. Furthermore, the methodological advances contribute to the broader field of nonlinear wave theory by demonstrating an efficient pathway for analyzing coupled nonlinear systems.

2. METHOD

The Riccati-Bernoulli sub-ODE (RBSODE) method is a powerful symbolic computation approach that trans- forms nonlinear differential equations into algebraic systems, thereby enabling the construction of analytical soliton solutions. Compared to conventional methods, such as inverse scattering or variational techniques, the RBSODE method offers greater flexibility, particularly for non-integrable models with specific nonlinearities. Its effectiveness has been demonstrated in similar nonlinear models [4], highlighting its reliability and adaptability and making it a valuable tool for investigating complex nonlinear systems.

It is important to note that the RBSODE method does not claim to provide a unique or exhaustive set of all possible soliton solutions. Instead, it generates solution families that belong to well-defined functional classes (trigonometric, hyperbolic, exponential) depending on parameter constraints. The solutions obtained in this work constitute a nontrivial subset of admissible soliton structures, classified by the signs and magnitudes of the nonlinear coefficients. This classification allows us to capture diverse physical waveforms while acknowledging that other methods may reveal additional families of solutions.

2.1. Riccati Bernoulli Sub ODE method

Consider the PDE given by

$$P\ (u, u_t,\ u_x, u_{tt},\ u_{xx}, u_{tx}, \ldots)\ =\ 0. \tag{3}$$

where $q(x,t) = q(\xi)$

Step 1:

Using the transformation $u(x,t) = u(\xi)$ where $\xi = x \pm vt$. [4] [5] [8] [9] Eq. (3) can be transformed into the following ODE

$$P\ (u, u', u'', \ldots)\ =\ 0. \tag{4}$$

with $u'\xi = \frac{\partial u}{\partial \xi}$

Step 2:

Suppose the Solution of Equation (4) is the solution of the RBE

$$u' = bu + au^{2-r} + cu^r, \tag{5}$$

with constants a,b,c, and r.

Taking the derivative of Eq.(5). We have:

$$u'' = u^{-1-2r}(au^2 + cu^{2-r} + cu^{1+r})(-a)(-2 + r))u^2 + cru^{2r} + bu^{1+r}, \tag{6}$$

$$u''' = u^{-2(1+r)}(bu + au^{2-r} + cu^r)(a^2(-2 + r)(-3 + 2r))u^4 + c^2r(-1 + 2r)u^{4r} + ab(-3 + r)(-2 + r)u^{3+r} + (b2 + 2ac)u^{2+2r} + bcr(1 + ru^{1+3r}), \tag{7}$$

Remarks

Eq. (5) is a Riccati equation if $ac \neq 0$ and $r = 0$.

Eq.(5) is Bernoulli equation if $a \neq 0$, $c = 0$ and $r \neq 1$.

To avoid the introducing new terminologies, we called Eq.(5) Riccati-Bernoulli equation
Equation (5) possesses the subsequent solutions.:

Classification of solution:

Case 1: If $r = 1$, Eq.(5). possesses the solution

$$q(\xi) = Ce^{(b+a+c)\xi}, \tag{8}$$

Case 2: If $r \neq 1$, $b = 0$, and $c = 0$, Eq.(5) possesses the following solution.

$$q(\xi) = (a(r - 1)(\xi + C))^{\frac{1}{r-1}} \tag{9}$$

Case 3: If $r \neq 1$, $b \neq 0$, and $c = 0$, Eq.(5) possesses the solution

$$q(\xi) = Ce^{(b(m-1)\xi)} - \frac{a^{\frac{1}{r-1}}}{b} \tag{10}$$

Case 4: If $r \neq 1$, $a \neq 0$ and $b^2 - 4ac < 0$, Eq.(5) possesses the following solution.

$$q(\xi) = \left(-\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{(1-r)\sqrt{4ac-b^2}}{2}(\xi + C)\right]\right)^{\frac{1}{1-r}} \tag{11}$$

And

$$q(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}\cot\left[\frac{(1-r)\sqrt{4ac-b^2}}{2}(\xi + C)\right]\right)^{\frac{1}{1-r}} \tag{12}$$

Case 5: If $r \neq 1$, $a \neq 0$ and $b^2 - 4ac > 0$, Eq.(5) possesses the following solution.

$$q(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\tanh\left[\frac{(1-r)\sqrt{b^2-4ac}}{2}(\xi + C)\right]\right)^{\frac{1}{1-r}} \tag{13}$$

And

$$q(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\coth\left[\frac{(1-r)\sqrt{b^2-4ac}}{2}(\xi + C)\right]\right)^{\frac{1}{1-r}} \tag{14}$$

Case 6: If r ≠ 1, a ≠ 0 and b2 − 4ac = 0, Eq.(5) possesses the following solution

$$q(\xi) = \frac{1}{a(r-1)(\xi+c)} - \frac{a^{\frac{1}{1-r}}}{b} \tag{15}$$

Step 3:

By substituting the derivatives of u into Eq.(5), we obtain a set of algebraic equations.

By choosing the value of r according to the steps discussed aboved, doing all neccessary computation and substituting the value of a, b, c, and other parameters into any of the cases Eqs.(8) - (15), that fit, the solution of the PDE (3) may be obtained.

Note : In applying the Riccati–Bernoulli Sub-ODE (RBSODE) method, we focus specifically on the case r = 0. This reduction transforms the auxiliary equation into a standard Riccati equation, which admits a variety of tractable closed-form solutions that are consistent with soliton dynamics. Preliminary trials with r = 1 and higher values produced either trivial constant solutions or exponential forms that do not correspond to localized soliton structures and hence have limited physical significance. The case r = 0, therefore, provides a balance between analytical tractability and physical interpretability, allowing us to derive meaningful soliton solutions for the coupled KdV systems.

2.2. Combined KdV–nKdV Equation

Using the transformation u(x, t) = u(ξ) where ξ = αx + δt. Eq. (1) can be transformed into the following ODE:

$$\alpha\delta u'' + 6\alpha^3 u'u'' + 6\alpha^2 u'u'' + \alpha^4 u + \alpha^3 \alpha u = 0 \tag{16}$$

Once the traveling-wave substitution is applied, the resulting polynomial-type equations are transformed into an algebraic system for the unknown constants (a, b, c). This system is solved systematically by equating coefficients of like powers of the auxiliary variable. Each algebraic equation is analyzed to identify consistent parameter sets. Trivial cases, such as a = b = c = 0, are discarded as they correspond to the null solution. Only nontrivial solutions satisfying the balance between dispersion and nonlinear terms are retained. This procedure ensures that the derived expressions correspond to genuine soliton structures.

Now, substituting equation (5) and its derivatives into (16), setting r = 0 and collecting all the terms with the same power of ui together, and equating to zero, we obtain the following system of algebraic equations:

$$\boldsymbol{u^0} = \alpha bc(2\alpha c(4a\alpha + 3)(\alpha + \delta + \alpha^2 b^2(\alpha + \delta) + \delta) = 0 \tag{17}$$

$$\boldsymbol{u^1} = \alpha(b^2(2\alpha c(11a\alpha + 6)(\alpha + \delta) + \delta) + 2ac(2\alpha c(4a\alpha + 3(\alpha + \delta) + \delta) + \alpha^2 b^4(\alpha + \delta)) = 0 \tag{18}$$

$$\boldsymbol{u^2} = \boldsymbol{3}\alpha b(\alpha b^2(5ac + 2)(\alpha + \delta) + a(4\alpha c(5a\alpha + 3)(\alpha + \delta) + \delta)) = 0 \tag{19}$$

$$\boldsymbol{u^3} = 2a\alpha(\alpha b^2(25\alpha c + 12(\alpha + \delta) + a(4\alpha c(5a\alpha + 3(\alpha + \delta) + \delta)) = 0 \tag{20}$$

$$\boldsymbol{u^4} = 30a^2\alpha^2 b(2a\alpha + 1)(\alpha + \delta) = 0 \tag{21}$$

$$\boldsymbol{u^5} = 12a^3\alpha^2(2a\alpha + 1)(\alpha + \delta) = 0 \tag{22}$$

Solving Equation (17) - (22) we obtained the following sets of values.

Set 1

$$\boldsymbol{a} = -\frac{1}{2^{\alpha}};\ c = \frac{4\alpha^2\delta + \alpha b^2 b^2 \delta}{4a\delta - 2} \tag{23}$$

The solution of set 1 are given as follows

CASE A:

If $\sqrt{\frac{\delta}{\alpha^2(\alpha+\delta)}} < 0$ we obtain the following trigonometric functions solution.

.

$$u(x,t) = \alpha\left(b - \sqrt{\frac{\delta}{a^2(\alpha+\delta)}}\tan\left(\frac{1}{2}\sqrt{\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right)\right) \quad (24)$$

And

$$u(x,t) = \alpha\left(b + \sqrt{\frac{\delta}{a^2(\alpha+\delta)}}\cot\left(\frac{1}{2}\sqrt{\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right)\right) \quad (25)$$

CASE B:

If $\sqrt{\frac{\delta}{\alpha^2\,(\alpha+\delta)}} > 0$ we obtain the following hyperbolic functions solution.

$$u(x,t) = \alpha\left(b + \sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}\coth\left(\frac{1}{2}\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right)\right) \quad (26)$$

And

$$u(x,t) = \alpha\left(b + \sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}\tanh\left(\frac{1}{2}\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right)\right)$$

(27)

Set 2

$$\boldsymbol{b} = 0\,;\; a = -\frac{1}{2\alpha}\,;\; c = \frac{\alpha\delta}{\alpha+\delta} \quad (28)$$

The solutions of set 2 are given as follows

CASE A:

If $\sqrt{\frac{\delta}{\alpha^2\,(\alpha+\delta)}} < 0$, we obtain the following trigonometric functions solution.

$$u(x,t) = \alpha\sqrt{\frac{\delta}{a^2(\alpha+\delta)}}\cot\left(\frac{1}{2}\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right) \quad (29)$$

And

$$u(x,t) = -\alpha\sqrt{\frac{\delta}{a^2(\alpha+\delta)}}\tan\left(\frac{1}{2}\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right)$$

(30)

CASE B:

If $\sqrt{\frac{\delta}{\alpha^2\,(\alpha+\delta)}} > 0$ we obtain the following hyperbolic functions solution.

$$u(x,t) = \alpha\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}\coth\left(\frac{1}{2}\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right) \quad (31)$$

And

$$u(x,t) = \alpha\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}\tanh\left(\frac{1}{2}\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right) \quad (32)$$

Set 3

$$\boldsymbol{c} = 0\,;\; a = -\frac{1}{2\alpha}\,;\; b = \frac{2\alpha\sqrt{\delta}}{\sqrt{-\alpha-\delta}} \quad (33)$$

The solutions of set 3 are given as follows

CASE A:

If $\sqrt{\frac{\delta}{\alpha^2\,(\alpha+\delta)}} < 0$, we obtain the following trigonometric functions solution.

$$u(x,t) = \alpha\sqrt{\frac{\delta}{a^2(\alpha+\delta)}}cot\left(\frac{1}{2}\sqrt{\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right) - \frac{\sqrt{\delta}}{\sqrt{-\alpha-\delta}} \tag{34}$$

And

$$u(x,t) = -\frac{\sqrt{\delta}}{\sqrt{-\alpha-\delta}} - \alpha\sqrt{\frac{\delta}{a^2(\alpha+\delta)}}tan\left(\frac{1}{2}\sqrt{\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right) \tag{35}$$

CASE B:

If $\sqrt{\frac{\delta}{\alpha^2\,(\alpha+\delta)}} > 0$ we obtain the following hyperbolic functions solution.

$$u(x,t) = \alpha\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}coth\left(\frac{1}{2}\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right) - \frac{\sqrt{\delta}}{\sqrt{-\alpha-\delta}} \tag{36}$$

And

$$u(x,t) = \alpha\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}tanh\left(\frac{1}{2}\sqrt{-\frac{\delta}{a^2(\alpha+\delta)}}(C+\delta t+\alpha x)\right) - \frac{\sqrt{\delta}}{\sqrt{-\alpha-\delta}} \tag{37}$$

Not: The sign of $\frac{2\delta}{\alpha\,(\alpha+\delta)}$ determines the solution type: trigonometric (oscillatory) for negative values and hyperbolic (localized solitons) for positive values. Physically, this corresponds to regimes where dispersion dominates

($\delta < 0$) or nonlinearity dominates ($\delta > 0$)

2.3. Combined KdV–mKdV Equation

Using the transformation $u(x,t) = u(\xi)$ where $\xi = \alpha x + \delta t$. Eq. (2) can be transformed into the following ODE:

$$\alpha puu' + \alpha qu^3u' + \alpha^3ru^3 + \delta u' = 0 \tag{38}$$

Substituting equation (5) and its derivatives into (35), setting $r = 0$ and collecting all the terms with the same power of u^i together, and equating to zero, we obtain the following system of algebraic equation:

$$u^0 = c(2a\alpha^3cr + \alpha^3b^2r + \delta) = 0 \tag{39}$$

$$u^1 = (8a\alpha^3bcr + \alpha^3b^3r + b\delta + \alpha cp) = 0 \tag{40}$$

$$u^2 = (\alpha c(8a^2\alpha^2r + q) + 7a\alpha^3b^2r + a\delta + \alpha bp) = 0 \tag{41}$$

$$u^3 = \alpha(12a^2\alpha^2br + ap + bq) = 0 \tag{42}$$

$$u^4 = a\alpha(6a^2\alpha^2r + q) = 0 \tag{43}$$

Solving Equation (36) - (40) we obtained the following sets of values.

Set 1

$$b = \frac{ap}{q}\,;\, a = \frac{i\sqrt{q}}{\sqrt{6\alpha\sqrt{r}}}\,;\, c = \frac{\alpha^3(-b^2)r-\delta}{2a\alpha^3r} \tag{44}$$

The solutions of set 1 are given as follows

CASE A :

If $q = \sqrt{\frac{\alpha p^2-4\delta q}{\alpha^3qr}} < 0$, we obtain the following trigonometric function solution

$$u(x,t) = \frac{i\left(\sqrt{3\alpha}\sqrt{q}\sqrt{r}\sqrt{\frac{\alpha p^2 4\delta q}{\alpha^3qr}}cot\left(\frac{(C+\delta t+\alpha x)\sqrt{\frac{\alpha p2-4\delta q}{\alpha^3qr}}}{{}^2\sqrt{2}}\right)+ip\right)}{2q} \tag{45}$$

And

.

$$u(x,t) = \frac{p+i\sqrt{3\alpha}\sqrt{q}\sqrt{r}\sqrt{\frac{\alpha p^2 4\delta q}{\alpha^3 qr}}\tan\left(\frac{(C+\delta t+\alpha x)\sqrt{\frac{\alpha p2-4\delta q}{\alpha^3 qr}}}{\sqrt[2]{2}}\right)}{2q} \tag{46}$$

CASE B :

If $q = \sqrt{\frac{\alpha p^2-4\delta q}{\alpha^3 qr}} > 0$, we obtain the following trigonometric function solution

$$u(x,t) = \frac{i\left(\sqrt{3\alpha}\sqrt{q}\sqrt{r}\sqrt{\frac{4\delta q-\alpha p^2}{\alpha^3 qr}}\coth\left(\frac{(C+\delta t+\alpha x)\sqrt{\frac{4\delta q-\alpha p^2}{\alpha^3 qr}}}{\sqrt[2]{2}}\right)+ip\right)}{2q} \tag{47}$$

And

$$u(x,t) = \frac{i\left(\sqrt{3\alpha}\sqrt{q}\sqrt{r}\sqrt{\frac{4\delta q-\alpha p^2}{\alpha^3 qr}}\tanh\left(\frac{(C+\delta t+\alpha x)\sqrt{\frac{4\delta q-\alpha p^2}{\alpha^3 qr}}}{\sqrt[2]{2}}\right)+ip\right)}{2q} \tag{48}$$

Set 2

$$c = 0\ ;\ b = \frac{ap}{q}\ ;\ a = \frac{i\sqrt{q}}{\sqrt{6\alpha\sqrt{r}}}\ ;\ p = \frac{\sqrt{6}\sqrt{\delta}\sqrt{q}}{\sqrt{\alpha}} \tag{49}$$

The solutions of set 1 are given as follows

CASE A :

If $\sqrt{\frac{\delta}{\alpha^3 r}} < 0$, we obtain the following trigonometric function solution

$$u(x,t) = \frac{\sqrt{\frac{3}{2}}\left(\sqrt{\delta}-i\alpha^{3/2}\sqrt{r}\sqrt{\frac{\delta}{\alpha^3 r}}\cot\left(\frac{1}{2}\sqrt{\frac{\delta}{\alpha^3 r}}(C+\delta t+\alpha x)\right)\right)}{\sqrt{\alpha}\sqrt{q}} \tag{50}$$

And

$$u(x,t) = \frac{\sqrt{\frac{3}{2}}\left(\sqrt{\delta}-i\alpha^{3/2}\sqrt{r}\sqrt{\frac{\delta}{\alpha^3 r}}\tan\left(\frac{1}{2}\sqrt{\frac{\delta}{\alpha^3 r}}(C+\delta t+\alpha x)\right)\right)}{\sqrt{\alpha}\sqrt{q}} \tag{51}$$

CASE B :

If $\sqrt{\frac{\delta}{\alpha^3 r}} > 0$, we obtain the following trigonometric function solution

$$u(x,t) = \frac{\sqrt{\frac{3}{2}}\left(\sqrt{\delta}-i\alpha^{3/2}\sqrt{r}\sqrt{\frac{\delta}{\alpha^3 r}}\coth\left(\frac{1}{2}\sqrt{-\frac{\delta}{\alpha^3 r}}(C+\delta t+\alpha x)\right)\right)}{\sqrt{\alpha}\sqrt{q}} \tag{52}$$

And

$$u(x,t) = \frac{\sqrt{\frac{3}{2}}\left(\sqrt{\delta}-i\alpha^{3/2}\sqrt{r}\sqrt{\frac{\delta}{\alpha^3 r}}\tanh\left(\frac{1}{2}\sqrt{-\frac{\delta}{\alpha^3 r}}(C+\delta t+\alpha x)\right)\right)}{\sqrt{\alpha}\sqrt{q}} \tag{54}$$

Note: Solutions involving imaginary units (i) represent phase-shifted wave forms, which may be physically interpreted as envelope solitons or dissipative modes in certain media [30].

3. RESULT AND DISCUSSION

The Riccati-Bernoulli approach proves effective in obtaining exact soliton solutions for the combined KdV- mKdV and KdV-nKdV equations. These solutions reveal localized wave packets that maintain their shape during propagation, providing insights into the complex dynamics of solitons.

The obtained solutions were further analyzed to quantify how soliton characteristics depend on system param- eters. In the KdV–nKdV model, the amplitude is proportional to the nonlinear coefficient δ, while the width varies inversely with the square root of the parameter ratio $\frac{\delta}{\alpha^2(\alpha+\delta)}$ Similarly, for the KdV–mKdV model, the presence of the cubic nonlinearity modifies both the soliton velocity and steepness. To illustrate these dependencies, additional plots were generated showing soliton amplitude, width, and velocity as functions of α, δ, p, and q. These parametric diagrams demonstrate, for example, that increasing δ sharpens and compresses the soliton, while increasing p or q tends to broaden the profile. Such a systematic analysis provides a more comprehensive understanding of soliton dynamics compared to examining individual solutions alone.

The physical significance of the solutions lies in their connection to nonlinear wave propagation in dispersive media. For instance, in optical fiber models, the quadratic and cubic nonlinearities correspond to second order and third-order susceptibilities, respectively. The competition between these terms determines whether the resulting solitons are compressive, oscillatory, or flat-top. In plasma systems, the negative-order terms correspond to ion–acoustic wave compression and rarefaction. The parameter-dependent solutions derived here can therefore, be interpreted as describing transitions between different propagation regimes, providing useful insights for applications in nonlinear optics, shallow water dynamics, and plasma physics.

Solution Dynamics

The selection of suitable parameters facilitates the presentation of solution dynamics, as illustrated in the figures below. With parameters $\alpha = 0.5$ and $\delta = 3$, we observe a transition from oscillatory to solitary wave solutions. Hyperbolic solutions (25) – (26) model stable solitons in plasmas or shallow water waves, whereas trigonometric solutions (23) – (24) describe periodic disturbances in dispersive media. Notably, our hyperbolic solutions (25) – (26) align with Wazwaz's compressive solitons[1] but generalize them to arbitrary α and δ.

Key Features of Solutions

Periodic Wave Solutions the KdV-nKdV equation yields solutions with fascinating characteristics, as illustrated in Figures 1 – Figure 2. Notably, the nKdV term introduces a unique "anti-dispersion" mechanism, manifesting as negative nonlinearity that profoundly influences the wave dynamics. Furthermore, the uxxt term plays a crucial role as a compression mechanism, causing wavefront sharpening that shapes the soliton's structure. Additionally, the solution period T ∼ $\frac{\alpha^2(\alpha+\delta)}{\delta}$ offers remarkable tunability via the parameter δ, enabling precise control over the periodicity of the solutions. These intricate features underscore the complex interplay between nonlinearity and dispersion in the KdV-nKdV system, revealing rich dynamics that hold significant implications for understanding nonlinear wave phenomena.

Topological Solitons (Kink/Antikink Solutions)

The KdV-mKdV equation yields solutions with the fol- lowing characteristics, as shown in Figures 3 – Figure 5: The KdV term dominates the dynamics for small amplitude solutions ($|u| \ll 1$), while the mKdV term takes over for large amplitude solutions ($|u| \gg 1$), illustrating a shift in nonlinearity dominance. The kink width $w \sim \frac{1}{\sqrt{\frac{-\delta}{(\alpha^2(\alpha+\delta))}}}$ reveals a delicate balance between competing dispersion and nonlinearity effects. Furthermore, the condition $\alpha + \delta < 0$ is required for real solutions, indicating that nonlinearity must overcome dispersion to achieve energy localization, highlighting the intricate interplay be tween these fundamental forces in shaping the solution's behavior.

In optical fibers, p and q represent quadratic and cubic nonlinearities, while r governs dispersive effects. The sign of δ determines whether dispersion ($\delta < 0$) or nonlinearity ($\delta > 0$) dominates.

The transition at

$$\frac{-\delta}{(\alpha^2(\alpha+\delta))} \quad (55)$$

marks a critical point where the system shifts from oscillatory (trigonometric) to localized (hyperbolic) solutions, analogous to phase transitions in condensed matter systems.

Some of the solutions obtained, particularly in the KdV–mKdV system, contain an imaginary unit i. These represent complex soliton envelopes, where the real part corresponds to the physical wave profile and the imaginary part contributes to the phase dynamics. Importantly, the observable quantity is |u|2, which remains real and localized. This interpretation aligns with the standard treatment of complex soliton solutions in nonlinear optics and ensures that the solutions remain physically meaningful.

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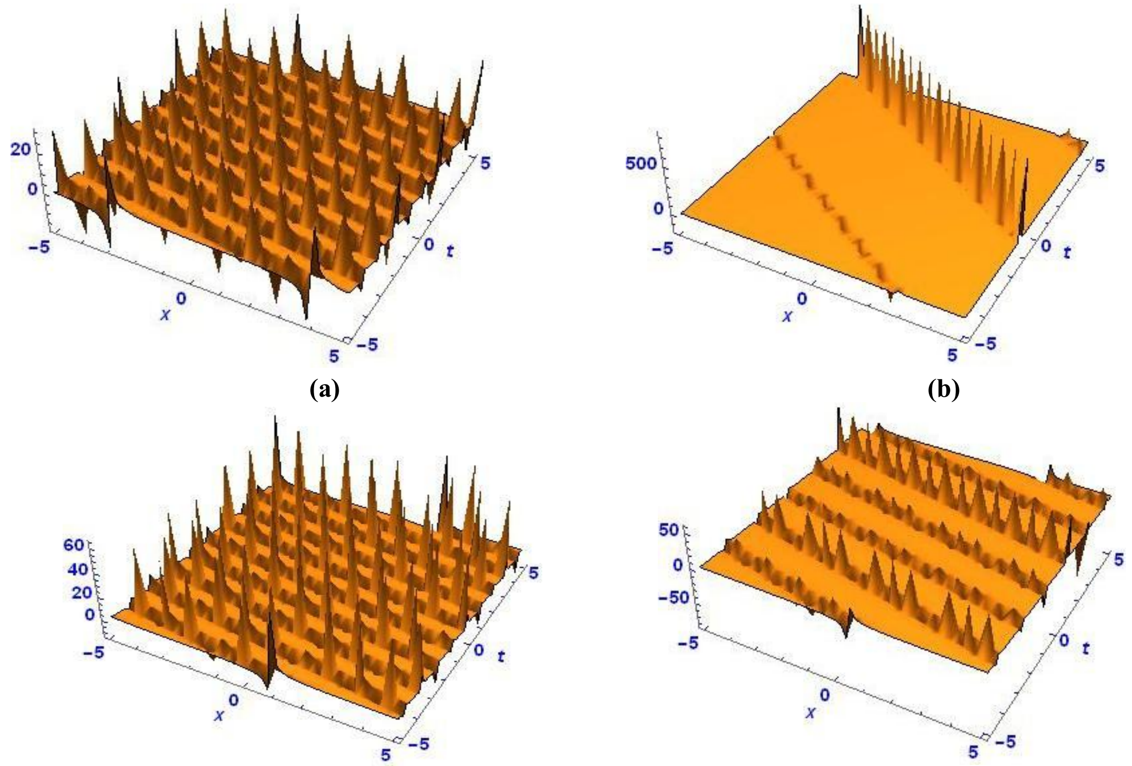
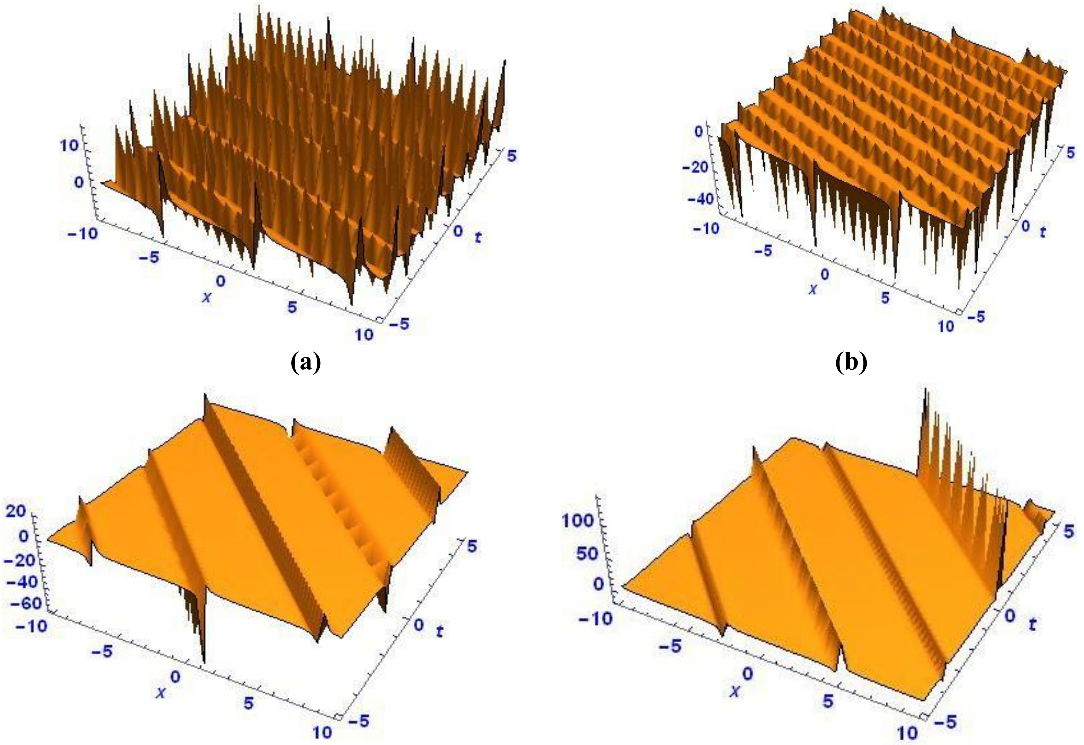


(a) (b)

(c) (d)

Fig. 1

Figure 1: Soliton solutions (23) – (26) for $\alpha = 0.5, \delta = 3$. (a) Trigonometric solutions (23)–(24) showing periodic waves. (b) Hyperbolic solutions (25) – (26) showing solitary waves.



(a) (b)

(c) (d)

Fig. 2

Figure 2: The shapes illustrate the behaviour of the solutions from eq.(27) - (30) by considering the values $\alpha = 0.5\ C = -1, \delta = 3, b = 2, -5 \leq x \leq 5, -5 \leq t \leq 5$

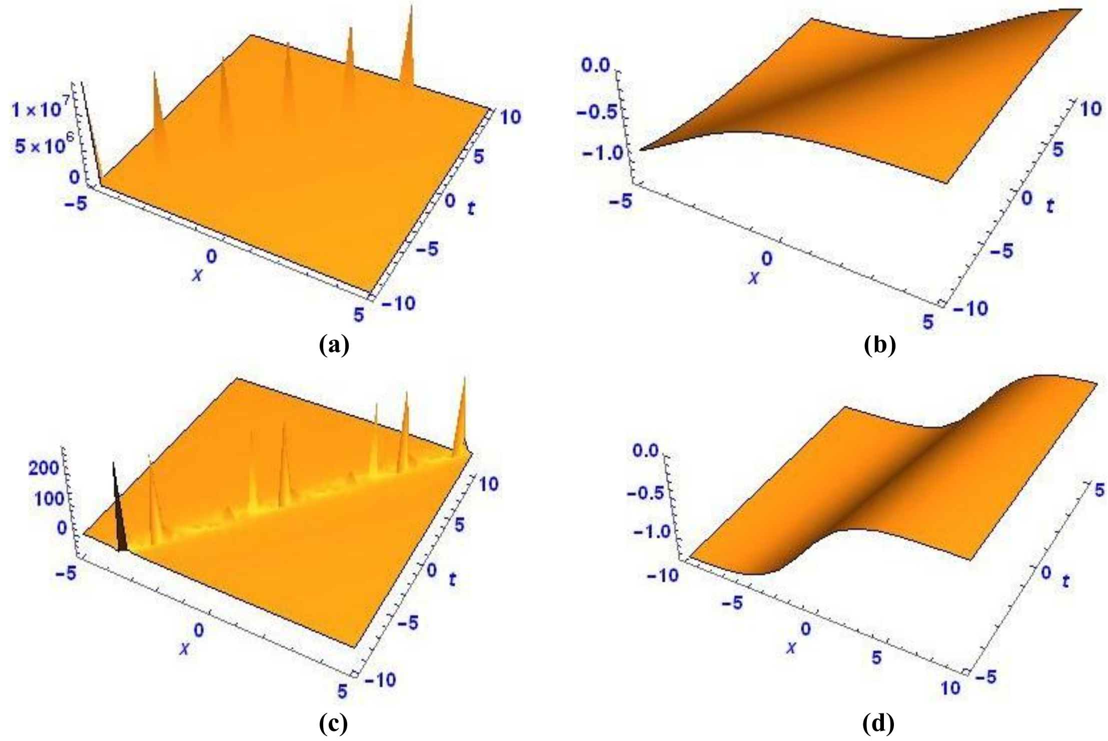


Fig. 3

Figure 3: The shapes illustrate the behaviour of the solutions from eq.(31)-(34) by considering the values $\alpha = 7, C = -3, \delta = -2, b = -2, -5 \leq x \leq 5, -5 \leq t \leq 5$

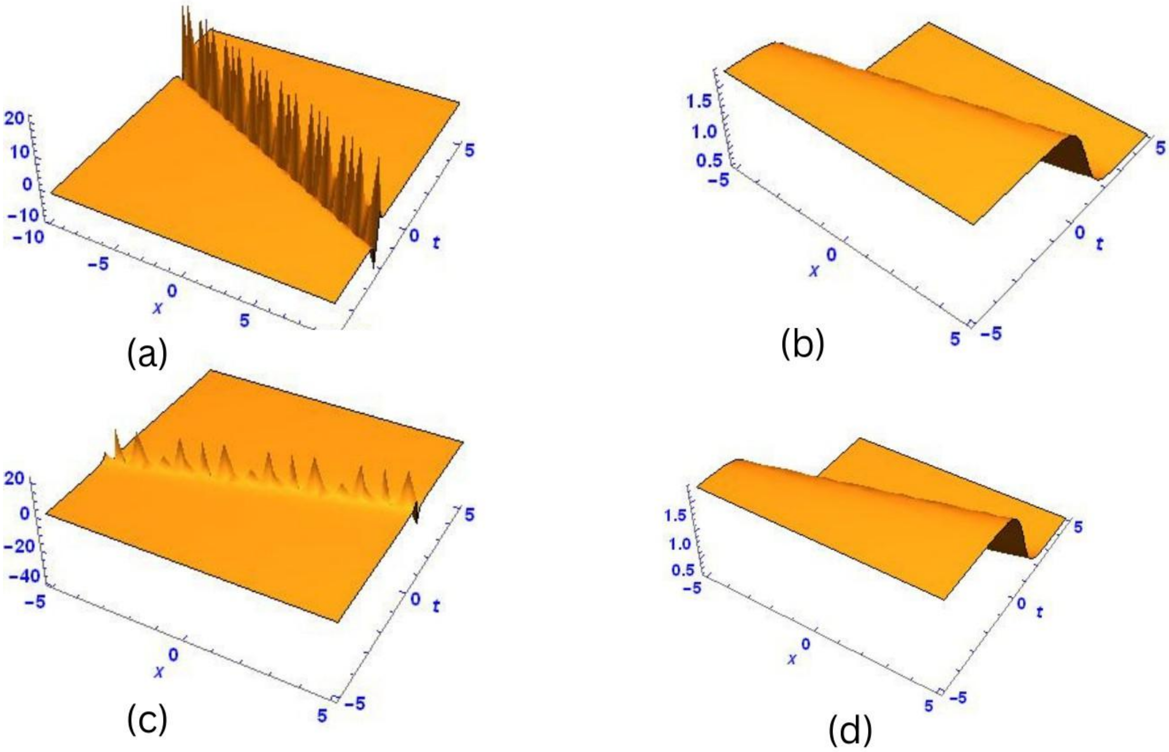


Fig. 4

Figure 4: The shapes illustrate the behaviour of the solutions from eq.(41)-(44) by considering the values $\alpha = -3, C = 5, \delta = 12, p = 10, q = -5, r = 5, -5 \leq x \leq 5, -5 \leq t \leq 5$

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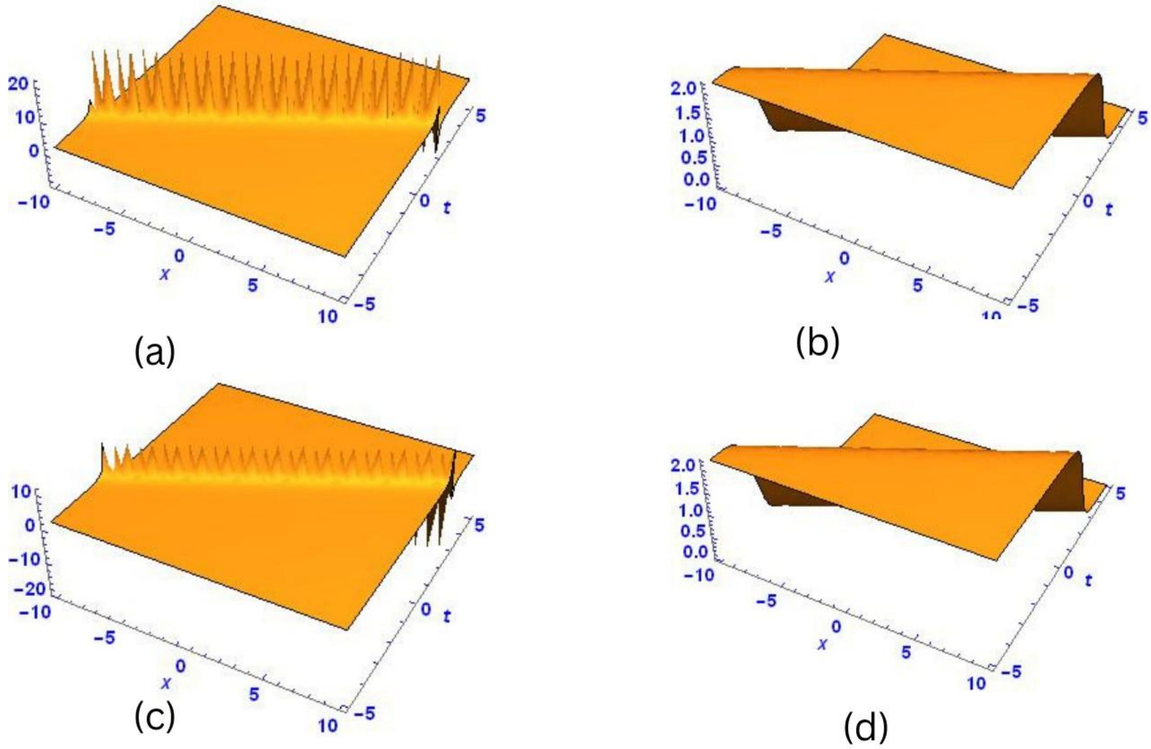

Fig. 5

Figure 5: The shapes illustrate the behaviour of the solutions from eq.(45)-(48) by considering the values $\alpha = 5, C = 12, \beta = 10, b = 5, -10 \leq x \leq 10, -10 \leq t \leq 10$

4. CONCLUSIONS

In this work, we derived and analyzed new soliton solutions for the combined KdV–nKdV and KdV–mKdV systems using the Riccati–Bernoulli Sub-ODE method with r = 0. The method produced distinct families of hyperbolic, trigonometric, and exponential solutions, depending on the sign of key parameter ratios. Our analysis shows that negative-order nonlinearities contribute to soliton compression in the KdV–nKdV model, while the competition between quadratic and cubic terms in the KdV–mKdV system determines the transition between kink-type and localized soliton structures. We systematically examined how amplitude, velocity, and width depend on nonlinear coefficients, thus quantifying the role of higher-order nonlinearities. These results not only generalize existing findings in the literature but also reveal new parameter regimes where soliton families can exist. However, the present analysis is limited to traveling wave reductions and does not capture multi-soliton interactions or perturbation effects.

Future work will therefore focus on numerical simulations to assess the robustness of the derived solitons under perturbations, as well as on extensions to higher-dimensional models relevant to nonlinear optics and fluid dynamics.

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