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Investigating Soliton-Wave Dynamics Using the Focusing Nonlinear Schro¨dinger Equation

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ABSTRACT

This research undertakes a comprehensive investigation of the optical soliton solutions of the Focusing Non- linear Schr¨odinger Equation (NLSE), a fundamental model describing the propagation of optical solitons in nonlinear media. We employ two versatile and efficient methods: the Ricatti-Bernoulli Sub Ordinary Differential Equation (RBSODE) method and the Bernoulli Sub Ordinary Differential Equation (BSODE) method. These methods enable us to derive a wide range of optical soliton solutions.We examine two distinct nonlinearities: the Kerr law nonlinearity and the quadratic-cubic nonlinearity. These nonlinearities are crucial in determining the behavior of optical solitons in various nonlinear optical media. Our analysis reveals that the derived soliton solutions exhibit distinct characteristics. Kerr nonlinearity supports sharper, narrower solitons, whereas quadratic-cubic nonlinearity yields broader profiles with enhanced stability. This study obtains soliton solutions of the NLSE with Kerr and QC nonlinearities using the RBSODE and BSODE methods, analyzes the qualitative differences in the obtained profiles, and examines the conservation laws characterizing the dynamics. The RBSODE and BSODE methods are chosen for their algebraic flexibility and their ability to handle the nonlinear ODEs derived from the traveling-wave reduction of the NLSE. Furthermore, we use the multiplier method to derive the conservation laws of the NLSE. These conservation laws provide valuable insights into the underlying dynamics of the optical solitons and have significant implications for the design and optimization of nonlinear optical systems. Our research contributes to the understanding of soliton behavior in nonlinear media, with potential applications in optical signal transmission and ultrafast laser propagation.

1. INTRODUCTION

The nonlinear Schr¨odinger equation is gaining prominence due to its extensive applications in areas such as fluid dynamics and optical technology. The focusing nonlinear Schr¨odinger equation is especially valuable for studying solitons in optical fibers. Solitons, stable waves that retain their form over long distances, are essential to optical communication systems.

The Focusing Nonlinear Schr¨odinger Equation is given by [1]:

$$iq_t + \frac{1}{2}q_{xx} + |q^2|q = 0 \tag{1}$$

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where $q = q(x,t)$ denotes the wave profile, where x represents a non-dimensional length within the material, and t signifies the elapsed time. The term |q|2q reflects the cubic nonlinearity in the refractive index of the material, which is defined as $n = n0 + n2|q|^2$, where n0 is the linear refractive index and n2 is the Kerr coefficient. The governing equation is utilized to model the propagation of a light beam through a medium with this refractive index. When the Kerr coefficient is negative $(n2 < 0)$, the nonlinearity is self-defocusing, leading to an expansion of the beam's width in the transverse x-direction over time. In contrast, a positive refractive index $(n2 > 0)$ causes the beam's width to contract, resulting in focusing. Numerous researchers have explored this model equation, as seen in studies like "The NLSE with Rogue periodic waves" [1], "Effective integration of ultra-elliptic solutions of the focusing NLSE" [2], "Soliton shielding in the focusing NLSE" [3], and "The stability spectrum for elliptic solutions to the focusing NLSE" [5], among others.

Despite the extensive literature on NLSE, challenges remain in constructing exact solutions for higher-order or generalized nonlinearities. Traditional methods, such as inverse scattering or Hirota's method, may not accommodate complex nonlinear terms like QC. Therefore, there is a need to apply direct algebraic approaches that can systematically handle these nonlinearities and yield physically meaningful solutions [26]. In this context, the NLSE, dispersion broadens optical pulses due to the frequency-dependence wave speed, while nonlinearity counters this effect by inducing pulse compression. A balance between these two competing effects gives rise to soliton structures—waveforms that maintain their shape during propagation. Kerr nonlinearity contributes a cubic term to the refractive index, while quadratic-cubic nonlinearity introduces higher-order corrections, enabling richer propagation dynamics in intense-field regimes [32].

The Riccati-Bernoulli sub-ODEs method (RBSODE) and the Bernoulli sub-ODEs method (BSODE) are symbolic computational techniques that allow exact reduction of nonlinear differential equations to algebraic systems, facilitating the analytical construction of soliton. Compared to inverse scattering or variational methods, these approaches offer greater algebraic flexibility, especially for non-integrable forms such as those with QC nonlinearity. These methods have been successfully applied to similar nonlinear models in [10][26], showcasing their reproducibility and ease of generalization. Specifically, the RBSODE method provides a symbolic algebraic framework to for reducing nonlinear ODEs to solvable forms via balance procedures, making it suitable for the QC-type NLSE, given its success in handling higher-order polynomial nonlinearities. We adopt it here for the systematic construction of soliton profiles, extending earlier applications by [6][41]. By using both RBSODE and BSODE techniques, this study provides a unique perspective on method sensitivity and solution structure, systematically contrasting Kerr and quadratic-cubic effects using exact analytical methods.

This study aims to construct exact optical soliton solutions for the NLSE with Kerr and QC nonlinearities using BSODE and RBSODE methods, and compare the qualitative impact of these nonlinearities on soliton structure and conservation properties. By doing so, we investigate the dynamics of solitons in nonlinear optical fibers, examining dispersion and nonlinearity to enhance data transmission efficiency, and we contribute to the theoretical understanding of nonlinear waves and their practical use in contemporary communication systems. Unlike earlier works [5][6][41] that focus largely on Kerr-type nonlinearities using perturbative or numerical techniques, this study explores mixed nonlinear regimes, such as QC, and examines conservation laws in detail alongside exact solutions, thereby enhancing physical interpretability and facilitating the development of innovative solutions in optics and physics.

2. METHOD

The Riccati-Bernoulli sub-ODE (RBSODE) and Bernoulli Sub ODE (BSODE) methods are symbolic computational techniques that allow exact reduction of nonlinear differential equations to algebraic systems, facilitating the analytical construction of solitons. Compared to inverse scattering or variational methods, these approaches offer greater algebraic flexibility, especially for non-integrable forms such as those with Quadratic-Cubic nonlinearity. The RBSODE method, in particular, offers a symbolic algebraic framework for reducing nonlinear ODEs to solvable forms via balance procedures, and its success in handling higher-order polynomial nonlinearities makes it suitable for the QC-type NLSE. These methods have been successfully applied to similar nonlinear models in [26][31], showcasing their reproducibility and ease of generalization.

2.1. Description of the proposed Methods

The Riccati-Bernoulli sub-ODE (RBSODE) and Bernoulli Sub ODE (BSODE) methods are symbolic computational techniques that allow exact reduction of nonlinear differential equations to algebraic systems, facilitating the analytical construction of solitons. Compared to inverse scattering or variational methods, these approaches offer greater algebraic flexibility, especially for non-integrable forms such as those with Quadratic-Cubic nonlinearity. The RBSODE method, in particular, offers a symbolic algebraic framework for reducing nonlinear ODEs to solvable forms via balance procedures, and its success in handling higher-order polynomial nonlinearities makes it suitable for the QC-type NLSE. These methods have been successfully applied to similar nonlinear models in [26][31], showcasing their reproducibility and ease of generalization.

2.1.1. Ricatti-Bernoulli Sub ODE Methods

Let us consider a PDE given as

$$P\left(q\frac{\partial q}{\partial t},\frac{\partial q}{\partial x},\frac{\partial^2 q}{\partial t^2},\frac{\partial^2 q}{\partial x^2},\dots\right)=0, \quad (2)$$

where $q(x,t) = q(\xi)$

Stage 1:

Using the conversion

$$q(x,t) = q(\xi) \times ei\phi(x,t), \quad (3)$$

where $\xi = \lambda(x \pm vt)$ and $\phi(x,t) = -k1x + \omega t + \theta$.

Eq. (2) can be rewritten as the accompanying following ODE.

$$P\,(q, q', q'', \dots) = 0, \quad (4)$$

with $q(\xi) = \frac{\partial q}{\partial \xi}$.

Stage 2:

Presuming that the solution to Eq.(4) satisfies the Riccati-Bernoulli Equation.

$$q' = bq + aq^{2-r} + cqr, \quad (5)$$

with constants a,b,c, and r.

Taking the derivative of Eq.(5). We have:

$$q'' = q - 1 - 2r(aq2 + cq2 - r + cq1 + r)(-a)(-2 + r))q2 + crq2r + bq1 + r \quad (6)$$

$$q''' = q - 2(1 + r)(bq + aq2 - r + cqr)(a2(-2 + r)(-3 + 2r))q4 + c2r(-1 + 2r)q4r + ab(-3 + r)(-2 + r)q3 + r + (b2 + 2ac)q2 + 2r + bcr(1 + rq1 + 3r), \quad (7)$$

Remarks

Eq. (5) is a Riccati equation if $ac \neq 0$ and $r = 0$.

Eq.(5) is Bernoulli equation if $a \neq 0$, $c = 0$ and $r \neq 1$.

To avoid the introducing new terminologies, we called Eq.(5) Riccati-Bernoulli equation Equation (5) possesses the subsequent solutions.:

Classification of solution:

Case 1: If $r = 1$, Eq. (5) possesses the solution

$$q(\xi) = Ce^{(b+a+c)\xi}, \quad (8)$$

Case 2: If $r \neq 1$, $b = 0$, and $c = 0$, Eq. (5) possesses the following solution.

$$q(\xi) = (a(r-1)(\xi + C))^{\frac{1}{r-1}} \quad (9)$$

.

Case 3: If $r \neq 1$, $b \neq 0$, and $c = 0$, Eq. (5) possesses the solution

$$q(\xi) = Ce^{(b(m-1)\xi)} - \frac{a^{\frac{1}{r-1}}}{b} \tag{10}$$

Case 4: If $r \neq 1$, $a \neq 0$ and $b^2 - 4ac < 0$, Eq. (5) possesses the following solution.

$$q(\xi) = \left(-\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{(1-r)\sqrt{4ac-b^2}}{2}(\xi + C)\right]\right)^{\frac{1}{1-r}} \tag{11}$$

and

$$q(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}\cot\left[\frac{(1-r)\sqrt{4ac-b^2}}{2}(\xi + C)\right]\right)^{\frac{1}{1-r}} \tag{12}$$

Case 5: If $r \neq 1$, $a \neq 0$ and $b^2 - 4ac > 0$, Eq. (5) possesses the following solution.

$$q(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\tanh\left[\frac{(1-r)\sqrt{b^2-4ac}}{2}(\xi + C)\right]\right)^{\frac{1}{1-r}} \tag{13}$$

and

$$q(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\coth\left[\frac{(1-r)\sqrt{b^2-4ac}}{2}(\xi + C)\right]\right)^{\frac{1}{1-r}} \tag{14}$$

Case 6: If $r \neq 1$, $a \neq 0$ and $b2 - 4ac = 0$, Eq. (5) possesses the following solution

$$q(\xi) = \frac{1}{a(r-1)(\xi+c)} - \frac{a^{\frac{1}{1-r}}}{b} \tag{15}$$

Stage 3:
When we substitute ψ and its derivatives into Equation (4), we derive a system of algebraic equations. By choosing the value of *r* according to the steps discussed above, doing all necessary computation, and substituting the value of *a, b, c,* and other parameters into any of the cases Eq.(8) - (15) that fit, the solution of the PDE (2) may be obtained.

2.1.2. Bernoulli Sub ODE Methods

Consider a PDE given as

$$P\left(q, \frac{\partial q}{\partial t}, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial t^2}, \frac{\partial^2 q}{\partial x^2}, \dots\right) = 0 \tag{16}$$

Stage 1:
Using the conversion

$$q(x,t) = q(\xi) \times e^{i\phi(x,t)} \tag{17}$$

where $\xi = \lambda(x \pm vt)$ and $\phi(x,t) = -k1x + \omega t + \theta$. Eq. (16) can be converted into the following ODE

$$P(q, q', q'', \dots) = 0 \tag{18}$$

with $q(\xi) = \frac{\partial q}{\partial \xi}$

Stage 2:
Assume that Equation (18) possesses a solution in the following form.

$$q(\xi) = \sum_{i=0}^{in} a_i g^i \tag{19}$$

where $G = G(\xi)$ satisfied the equation

$$G^{i} + \lambda G = \mu G^{2} \quad (20)$$

a_i are constants and $\mu \neq 0$, $\lambda \neq 0$.

The solution to Equation (20) is a particular form of the Bernoulli equation, which can be expressed as follows

$$G = -\frac{\lambda}{2\mu}\left(Tanh[\frac{\lambda\xi}{2}] - 1\right) \quad (21)$$

and

$$G = -\frac{\lambda}{2\mu}(Coth[\frac{\lambda\xi}{2}] - 1 \quad (22)$$

Stage 3:

The positive integer m is determined by equating the highest-order derivatives with the highest order non-linear term present in equation (18). The balancing formula is given as

$$D\left(\frac{d^{a}u}{d\xi^{a}}\right) = m + a, \quad D\left(u^{b}\left(\frac{d^{a}u}{d\xi^{a}}\right)^{c}\right) = bm + c(\mathrm{n+a}) \quad (23)$$

Stage 4:

By replacing equation (19) into (18), applying (20), and consolidating terms with the same power of $G(\xi)$, we establish a set of algebraic equations. Setting each coefficient of G^i to zero leads us to a system of algebraic equations. Solving this system yields the values of a_i and other associated parameters. Lastly, by putting the values of a_i and the associated parameters ~~in~~ into equation (19), we obtain the solution to equation (16).

2.2. Applications of the Methods

2.2.1. Application of the RBSODE Method to F-NLSE

a. Kerr Law non-linearity

Regarding the Kerr law nonlinearity $F(q) = q$.

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0. \quad (24)$$

By employing equation (3) in equation (24) and segregating the real and imaginary components of the equation, we arrive at the following. The imaginary component is:

$$v = k \quad (25)$$

The real component is:

$$k^2 q - q'' + 2\omega q - 2q^3 = 0 \quad (26)$$

By inserting equation (5) along with its derivatives into equation (26) and assigning $m = 0$, we obtain an overdetermined equation. To resolve this, we collect terms with identical exponents of q^i and equate them to zero, thereby deriving the following system of algebraic equations.

$$q^0 : -bc = 0, \quad (27)$$

$$q^1 : (-2ac - b^2 + k^2 + 2\omega) = 0 \quad (28)$$

$$q^2 : 3ab = 0 \quad (29)$$

$$q^3 : -2(a^2 + 1) = 0 \quad (30)$$

.

From solving Eq. (27) - (30), we obtained the following values:

$a = i; b = 0; v = k; c = \frac{1}{2}(-ak^2 - 2a\omega); \; \xi = x + vt$

The solution of the obtained values is given as follows:

CASE A:
If $k, \omega < 0$, we acquire the subsequent solutions in terms of trigonometric functions

$$q_{1,1}(x,t) = i\sqrt{\frac{k^2}{2} + \omega}\cot\left(\sqrt{\frac{k^2}{2} + \omega}(C + kt + x)\right) \times e^{i(\theta - kx + t\omega)} \quad (31)$$

and

$$q_{1,2}(x,t) = -i\sqrt{\frac{k^2}{2} + \omega}\tan\left(\sqrt{\frac{k^2}{2} + \omega}(C + kt + x)\right) \times e^{i(\theta - kx + t\omega)} \quad (32)$$

CASE B:
If $k, \omega, > 0$, we obtain the following solutions in terms of hyperbolic functions.

$$q_{1,3}(x,t) = i\sqrt{-\frac{k^2}{2} - \omega}\coth\left(\sqrt{-\frac{k^2}{2} - \omega}(C + kt + x)\right) \times e^{i(\theta - kx + t\omega)} \quad (33)$$

and

$$q_{1,4}(x,t) = i\sqrt{-\frac{k^2}{2} - \omega}\tanh\left(\sqrt{-\frac{k^2}{2} - \omega}(C + kt + x)\right) \times e^{i(\theta - kx + t\omega)} \quad (34)$$

b. Quadratic Cubic law non-linearity

Regarding Quadratic Cubic law non-linearity, $F(q) = \sqrt{q} + q$, where the coefficient of nonlinearity is 1.

$$iq_t + \frac{1}{2}q_{xx} + (|q| + |q|^2)q = 0. \quad (35)$$

By employing equation (3) in equation (35) and segregating the real and imaginary components of the equation, we arrive at the following.
The imaginary component is:

$$v = k. \quad (36)$$

The real component is:

$$k^2q - q'' + 2\omega q - 2q^3 - 2q^2 = 0 \quad (37)$$

By substituting equation (5) and its derivatives into equation (37) with $m = 0$, we obtain an overdetermined equation. To resolve this, we collect terms with identical exponents of q^i and set them to zero. This process leads us to formulate the following system of algebraic equations.

$$q^0 : -bc = 0, \quad (38)$$

$$q^1 : (-2ac - b^2 + k^2 + 2\omega) = 0, \quad (39)$$

$$q^2 : (-3ab - 2) = 0, \quad (40)$$

$$q^3 : -2(a^2 + 1) = 0, \quad (41)$$

From solving Eq. (38) - (41), we obtained the following Set of values:

$a = i; c = 0; b = \frac{2a}{3}; v = k; k = \frac{1}{3}\sqrt{2}\sqrt{-9\omega - 2}; \xi = x + vt$

The solution of the obtained values is given as follows:
CASE A:
If k, ω < 0, we obtain the following solutions in terms of trigonometric functions.

$$q_{2,1}(x,t) = \frac{1}{3}i\left(cot\left(\frac{1}{3}\left(C + \frac{1}{3}\sqrt{2}t\sqrt{-9\omega - 2} + x\right)\right) + i\right) \times e^{i(\theta + t\omega - \frac{1}{3}\sqrt{2}x\sqrt{-9\omega-2})} \quad (42)$$

and

$$q_{2,2}(x,t) = -\frac{1}{3}i\left(tan\left(\frac{1}{3}\left(C + \frac{1}{3}\sqrt{2}t\sqrt{-9\omega - 2} + x\right)\right) - i\right) \times e^{i(\theta + t\omega - \frac{1}{3}\sqrt{2}x\sqrt{-9\omega-2})} \quad (43)$$

CASE B:
If *k, ω, > 0*, we obtain the following solutions in terms of hyperbolic functions.

$$q_{2,3}(x,t) = \frac{1}{3}i\left(cot\left(\frac{1}{3}\left(C + \frac{1}{3}\sqrt{2}t\sqrt{-9\omega - 2} + x\right)\right) + i\right) \times e^{i(\theta + t\omega - \frac{1}{3}\sqrt{2}x\sqrt{-9\omega-2})} \quad (44)$$

and

$$q_{2,4}(x,t) = -\frac{1}{3}i\left(tan\left(\frac{1}{3}\left(C + \frac{1}{3}\sqrt{2}t\sqrt{-9\omega - 2} + x\right)\right) - i\right) \times e^{i(\theta + t\omega - \frac{1}{3}\sqrt{2}x\sqrt{-9\omega-2})} \quad (45)$$

2.2.2. Application of BSODE Method to F-NLSE

a. Kerr Law non-linearity

Regarding the Kerr law nonlinearity, $F(q) = q$.

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0. \quad (46)$$

By substituting equation (17) into equation (46) and then separating the equation into its real and imaginary parts, we obtain the following results. The imaginary component is:

$$v = k \quad (47)$$

The real component is:

$$k^2 q - q'' + 2\omega q - 2q^3 = 0 \quad (48)$$

By equating the coefficients of q^3 and q'' in equation (48), we find $m = 1$. Plugging in $m = 1$ into equation (19), we get the following.

$$q(\xi) = a_0 + a_1 G(\xi)\ . \quad (49)$$

Where a_0 and a_1 are constants to be determined. By substituting equation (49) and its derivatives into equation (48), we obtain an overdetermined expression. After gathering the terms of G^i and performing all the required calculations, we obtain the following results:

$$G^0 : a_0(-2a_0^2 + k^2 + 2\omega) = 0, \quad (50)$$

$$G^1 : a_1(-6a_0^2 + k^2 - \lambda^2 + 2\omega) = 0, \quad (51)$$

$$G^2 : 3a_1(\lambda\mu - 2a_0 a_1) = 0, \quad (52)$$

$$G^3 : -2a_1(a_1^2 + \mu^2) = 0, \quad (53)$$

From Eq. (50) - (53), we get the following Set of values.

$$a_0 = \frac{i\lambda}{2};\ v = k;\ a_1 = -\frac{2a_0\mu}{\lambda};\ k = \frac{\sqrt{-\lambda^2 - 4\omega}}{\sqrt{2}};\ \xi = tv + x$$

The solution of the obtained values is given as follows:

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$$q_{4,1}(x,t) = \frac{1}{2} i\lambda \coth\left(\frac{1}{2}\lambda\left(t\sqrt{-\frac{\lambda^2}{2} - 2\omega} + x\right)\right) \times e^{i\left(\theta + t\omega - \frac{x\sqrt{-\lambda^2-4\omega}}{\sqrt{2}}\right)} \tag{54}$$

and

$$q_{4,2}(x,t) = \frac{1}{2} i\lambda \tanh\left(\frac{1}{2}\lambda\left(t\sqrt{-\frac{\lambda^2}{2} - 2\omega} + x\right)\right) \times e^{i\left(\theta + t\omega - \frac{x\sqrt{-\lambda^2-4\omega}}{\sqrt{2}}\right)} \tag{55}$$

b. Quadratic Cubic law non-linearity

For Quadratic Cubic law non-linearity, $F\ (q) = \sqrt{q} + q.$

$$iq_t + \frac{1}{2} q_{xx} + (|q| + |q|^2)q = 0. \tag{56}$$

Employing equation (17) in equation (56) and segregating the real and imaginary components of the equation, we arrive at the following. The imaginary component is:

$$v = k \tag{57}$$

The real component is:

$$k^2 q - q'' + 2\omega q - 2q^3 - 2q^2 = 0 \tag{58}$$

By equating the coefficients of q^3 and q'' in equation (58), we find $m = 1$. Plugging in $m = 1$ into equation (19), we get the following.

$$q(\xi) = a_0 + a_1 G(\xi)\ . \tag{59}$$

where a_0 and a_1 are constants to be determined. By substituting equation (59) and its derivatives into equation (58), we obtain an overdetermined system. After gathering the terms of G^i and performing all the required calculations, we get:

$$G^0 : a_0(-2a_0^2 + 2a_0 + k^2 + 2\omega) = 0, \tag{60}$$

$$G^1 : a_1(-6a_0^2 - 2a_0 + k^2 - \lambda^2 + 2\omega) = 0, \tag{61}$$

$$G^2 : a_1(3\lambda\mu - 2(3a_0 + 1)a_1) = 0, \tag{62}$$

$$G^3 : -2a_1(a_1^2 + \mu^2) = 0, \tag{63}$$

From solving Eq. (60) - (63), we obtained the following set values.

Set 1:

$$k = \frac{1}{3}\sqrt{2}\sqrt{-9\omega - 2};\ a_0 = \frac{1}{2} 2k^2 + \lambda^2 + 4\omega;\ a_1 = -\frac{3\lambda\mu 6k^2 + 3\lambda^2 + 12\omega + 1}{9\lambda^2 - 2};\ \lambda = \frac{2i}{3};\ v = k$$

The solution of the obtained values is given as follows:

$$q_{5,1}(x,t) = \frac{1}{3} i\left(\cot\left(\frac{1}{9}\left(\sqrt{2}t\sqrt{-9\omega - 2} + 3x\right)\right) + i\right) \times e^{i(\theta + t\omega - \frac{1}{3}\sqrt{2}x\sqrt{-9\omega-2})} \tag{64}$$

and

$$q_{5,2}(x,t) = -\frac{1}{3} i\left(\tan\left(\frac{1}{9}\left(\sqrt{2}t\sqrt{-9\omega - 2} + 3x\right)\right) - i\right) \times e^{i(\theta + t\omega - \frac{1}{3}\sqrt{2}x\sqrt{-9\omega-2})} \tag{65}$$

Set 2:

$$a_0 = 0;\ \lambda = \frac{2i}{3};\ v = k; a_1 = \frac{3\lambda\mu}{2};\ k = \sqrt{\lambda^2 - 2\omega}$$

The solution of the obtained values is given as follows:

$$q_{6,1}(x,t) = -\frac{1}{3}i\left(\cot\left(\frac{1}{9}\left(\sqrt{2}t\sqrt{-9\omega-2}+3x\right)\right)-i\right)\times e^{i\left(\theta+t\omega-x\sqrt{-2\omega-\frac{4}{9}}\right)} \quad (66)$$

and

$$q_{6,2}(x,t) = \frac{1}{3}i\left(\tan\left(\frac{1}{3}\left(t\sqrt{-2\omega-\frac{4}{9}}+x\right)\right)+i\right)\times e^{i\left(\theta+t\omega-x\sqrt{-2\omega-\frac{4}{9}}\right)} \quad (67)$$

2.3. Analysis of Conservation Laws

In this section, we analyse the conservation laws (CLs) of the focusing nonlinear Schrödinger equation using the direct method, which employs multipliers. To do this, we will start by converting the equation into a system of nonlinear partial differential equations (NLPDEs) through the following transformation [22]:

$$q(x,t) = u(x,t) + iv(x,t), \quad (68)$$

where *u(x,t)* and *v(x,t)* are functions. Substituting Eq. (68) into Eq. (1) and separating the real and imaginary parts, we get:

$$-v_t + \frac{1}{2}u_{xx} + (u^2+v^2)u = 0 \quad (69)$$

$$u_t + \frac{1}{2}v_{xx} + (u^2+v^2)v = 0$$

Next, we will provide a concise overview of the methods and subsequently apply these principles to the generalized unstable nonlinear Schrödinger equation.

2.3.1. Conservation Laws Using the Multiplier Approach

Let $x = (x1, x2, \dots, xn)$ denote *n* independent variables, and $u = (u^1, u^2, \dots, u^m)$ represent *m* dependent variables. We consider a system of *r* PDEs of k^{th}-order described by [22]:

$$R\alpha[u] = R\alpha\big(x, u, u(1), u(2), \dots, u(k)\big), \ a = 1, 2, \dots, r \quad (70)$$

where $u_{(1)} = \{u^{\alpha}_{(ij)}\}, \{u^{\alpha}_{(i)}\} = \frac{\partial u^{\alpha}_{i}}{\partial x_i}, \{u^{\alpha}_{(ij)}\} = \frac{\partial^2 u^{\alpha}}{\partial x_i \partial x_j}$, ... Let $(M2, M2, \dots, MN)$ denote arbitrary functions of the independent variables *x*, and denote partial derivatives $\frac{\partial}{\partial x_i}$ by subscripts *i* [22], i.e.,

$$M^{\sigma}_{i} = \frac{\partial M^{\sigma}}{\partial x_i}, M^{\sigma}_{ij} = \frac{\partial^2 M^{\sigma}}{\partial x_i x_j}$$

1. The local conservation laws multiplier is given in the form
2. $C_i = \frac{\partial}{\partial x_i} + u^{\alpha}_{ij}\frac{\partial}{\partial u^{\alpha}_{i}} + u^{\alpha}_{ijk}\frac{\partial}{\partial u^{\alpha}_{jk}} + \cdots$..
3. Multipliers for system Eq. (70) are a set of functions $\{\Psi\alpha[M\,]\}$ satisfying:

$$\Psi\alpha[M\,]R\alpha[M\,] = CiNi[M\,], \quad (73)$$

For some functions $Ni[M\,]. If\ M\sigma = M\sigma(x)$ is the solution of PDE Eq. (70), from Eq. (101), we obtain the CLs [22]

$$CiNi[M\,] = 0 \quad (74)$$

of Eq. (98) and for each $i, Ni[M\,]$ is a flux.

.

4. The standard Euler operators with respect to the differential function M^j and the derivatives $M^i, M^j_{i_1 i_2}, .,$ are defined by:

5. $$E^j_M = \frac{\partial}{\partial M^i} - C_i \frac{\partial}{\partial M^j_i} + \ldots + (-1)^s C_{i_1} \ldots C_{i_s} \frac{\partial}{\partial M^j_{i_1 \ldots i_s}} \tag{75}$$

For each $j = 1, 2, \ldots, m. \{\Psi^\alpha[M]\}$ yields a set of multipliers for the CLs of Eq. (70) if each Euler operator in Eq. (70) annihilates the left side of Eq. (73):

$$E^j_M(\Psi^\alpha[M] R_\alpha[M] = 0,\ j = 1, \ldots, n \tag{76}$$

for arbitrary $M, Mi, Mij, \ldots.$

2.3.2. Application of the Multiplier Approach to Focusing on NLSE

Substituting equation (69) in equation (75) multiplied by Ψ yield

$$E_M\left(\Psi^1\left(-u_t + \frac{1}{2}u_{xx} + u^2 + v^2 u\right) + \Psi^2\left(v_t + \frac{1}{2}v_{xx} + u^2 + v^2 v\right)\right) \tag{77}$$

After expansion with respect to different combinations of derivatives of *u* and *v*, we yield the following overdetermined system for the multipliers Ψ^1 and Ψ^2

$$\Psi^2_{x,x} = 0, \Psi^1_{v_x,v_x} = 0, \Psi^1_t = \frac{\Psi^2_x v_x}{v}, \Psi^2_t = \frac{\Psi^2_x u_x}{v}, \Psi^1_t = \frac{\Psi^2_x u}{v}, \Psi^1_u = \frac{-\Psi^1_{v_x} u_t v_x + \Psi^1_{v_x} u_t v_x + \Psi^2 v_t + \Psi^1 u_t}{u u_t + v v_t}$$
$$\Psi^2_u = 0, \Psi^1_v = 0, \Psi^2_v = \frac{-\Psi^1_{v_x} u_t v_x + \Psi^1_{v_x} u_x v_t + \Psi^2 v_t + \Psi^1 u_t}{u u_t + v v_t}, \Psi^1_{v_t} = 0, \Psi^2_{u_x}$$
$$= \frac{-\Psi^1_{v_x} u_x u + \Psi^1_{v_x} v_x v + \Psi^2 u + \Psi^1 v}{u u_t + v v_t},$$
$$\Psi^1_{v_t} = \frac{-\Psi^1_{v_x} u_x u + \Psi^1_{v_x} v_x v + \Psi^2 u + \Psi^1 v}{u u_t + v v_t}, \Psi^2_{v_t} = 0, \Psi^1_{u_x} = 0, \Psi^2_{u_x} = -\Psi^1_{v_x} = 0, \Psi^2_{v_x} = 0 \tag{78}$$

By solving the system of partial differential equations described in Eq. (78), we derive the following zeroth- order multipliers for the model: $\Psi^1(x, t, u, v, ut, vt, ux, vx)$ and $\Psi^2(x, t, u, v, ut, vt, ux, vx)$, which are expressed as follows:

$$\Psi^1 = (D_1 x + D_2)u + (-D_1 - D_3)v_x - D_4 v_t,$$

$$\Psi^2 = (D_1 x + D_2)v + (-D_1 - D_3)u_x - D_4 u_t \tag{79}$$

where D_1, D_2, D_3 and D_4 are constants. Using Eq. (98) and Eq. (78), we obtained the following flux equations:

$$\mathbf{flux_t} = \frac{-u^2 D_4}{4} - \frac{ux^2 D_4}{2} + \frac{u^2 D_1 x}{2} + \frac{u^2 D_2}{2} + \frac{v_x v^3 D_1 t^2}{2} + \frac{u v^2 D_4}{2} - v^3 v_x D_3 t - u v_x D_1 t - u v_x D_1 t - u v_x D_3 + \frac{ux^2 D_4}{4} \tag{80}$$
$$\mathbf{flux_x} = \frac{-ux^2 D_3}{4} - \frac{ux^2 D_3}{4} + \frac{u v_x D_2}{2} + \frac{u^2 D_2}{2} + \frac{u^3 D_3}{4} + u v_x D_3 - \frac{v_t v_x D_4}{2} - \frac{ux^2}{4} D_1 t - \frac{u^2 D_1 t}{4} - \frac{u_x v D_2}{2} - \frac{u_t u_x D_4}{2}$$
$$- \frac{v^2 u^2 D_3}{2} + \frac{u u D_1}{2} + \frac{v_t v^3 D_1 t^2}{2} u v_t D_1 t + \frac{u v_x D_1 x}{2} - \frac{u_x v D_1 x}{2} + v^3 v_t D_3 t - \frac{v^2 u^2 D_1 t}{2} \tag{81}$$

From the obtained flux, we get the following conserved vectors:

– If $D_1 = 1, D_2, D_3 = 0$ and $D_4 = 0$ then we have the following conserved vectors:

$$\Psi^1 = v_x, \qquad \Psi^2 = u,$$
$$Z^t = \frac{u^2 x}{2} - \frac{v_x v^3 t^2}{2} - u v_x t \tag{82}$$

$$Z^t = \frac{-ux^2}{4} t - \frac{u^4 t}{4} - \frac{uv}{2} - \frac{vx^2 t}{4} + \frac{v_t v^3 t^2}{2} + u v_t t + \frac{u v_x x}{2} - \frac{u_x v_x}{2} - \frac{v^2 u^2 t}{2} \tag{83}$$

– If $D_1 = 0$, $D_2, = 1$, $D_3 = 0$ and $D_4 = 0$ then we have the following conserved vectors:

$$Z^t = \frac{u^2}{2} \tag{84}$$

$$Z^x = \frac{uv_x}{2} - \frac{u_x v}{2} \tag{85}$$

– If $D_1 = 0$, $D_2, = 0$, $D_3 = 1$ and $D_4 = 0$ then we have the following conserved vectors:

$$Z^t = -v^3 v_x t - uv_x \tag{86}$$

$$Z^t = -\frac{-vx^2}{4} - \frac{vx^2}{4} - \frac{u^4}{4} + uu_t - \frac{v^2u^2}{2} + v^3 v_t t \tag{87}$$

– If $D_1 = 0$, $D_2, = 0$, $D_3 = 0$ and $D_4 = 1$ then we have the following conserved vectors:

$$Z^t = \frac{-u^2}{4} - \frac{v^2u^2}{2} + \frac{vx^2}{2} - \frac{ux^2}{4} \tag{88}$$

$$Z^x = \frac{1}{2} u_t v_x - \frac{1}{2} u_v u_t \tag{89}$$

3. RESULTS AND DISCUSSION

From an optical physics standpoint, Kerr nonlinearity produces highly localized solitons that are more sensitive to input power, whereas quadratic-cubic nonlinearity leads to broader, more stable waveforms that resist perturbations. These distinctions are crucial for designing robust transmission lines in nonlinear fiber optics, where trade-offs between pulse shape, bandwidth, and power thresholds are critical.

Figure 1 displays the bright soliton profile under Kerr nonlinearity, illustrating the localized amplitude decay typical of cubic interactions. In contrast, Figure 2 shows that QC-based solitons display asymmetric widths and slight amplitude shifts, indicating that the additional cubic-quadratic interaction affects pulse compression. These findings align with prior reports by [41] on pulse broadening in QC media, though our profiles show stronger peak retention. This analysis aids in designing fiber systems with nonlinear compensation mechanisms, particularly in media where both quadratic and cubic terms coexist, such as photorefractive or birefringent media.

3.1. Visual Representation of Results

This section presents the results in 3D graphs, offering a comprehensive understanding of the solutions' physical behavior. Figures 1 and Figure 2 illustrate the solutions derived using the Ricatti-Bernoulli Sub ODE method under Kerr law and Quadratic-Cubic law nonlinearities, respectively. These visualizations facilitate the identification of patterns and trends in the solutions, enabling a deeper understanding of the underlying dynamics.

Figure 1 shows bell-shaped solitary waves that rise from a zero background, indicating localized wave packets; specifically, a bright soliton emerges with a distinct peak with Symmetric decay on both ends. In contrast, Figure 2 depicts dark solitons, characterized by localized dips or notches in amplitude against a constant (non-zero) background, typically in a continuous wave.

Similarly, Figures 4 and Figure 5 depict the physical behaviors of the solutions obtained with the Bernoulli Sub-ODE method under Kerr law and Quadratic-Cubic law nonlinearities, respectively. These graphical representations provide valuable insights into the solutions' characteristics, enabling a more nuanced interpretation of the results. Figure 4 specifically exhibits periodic solitons or cnoidal wave patterns with localized peaks and troughs that appear to maintain their form. In contrast, Figures 5 show dark soliton dips in amplitude, featuring localized dips that propagate without significantly changing shape.

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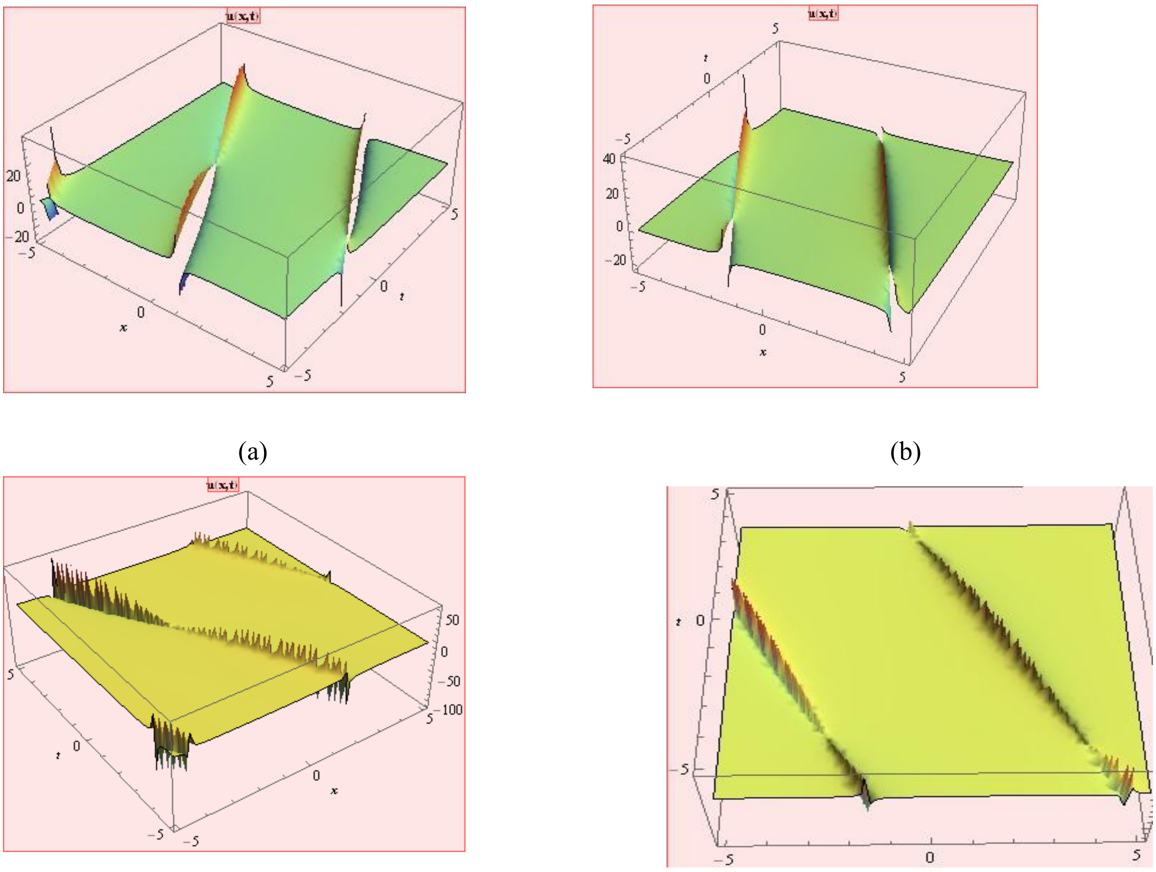
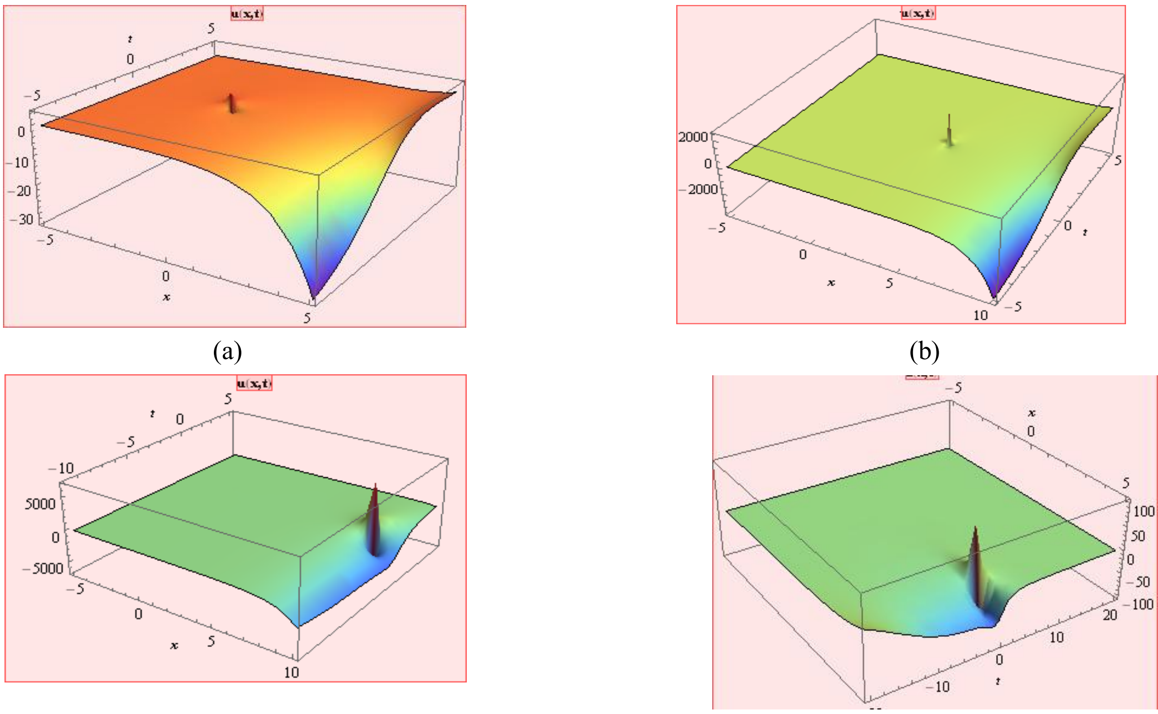


(a) (b)

(c) (d)

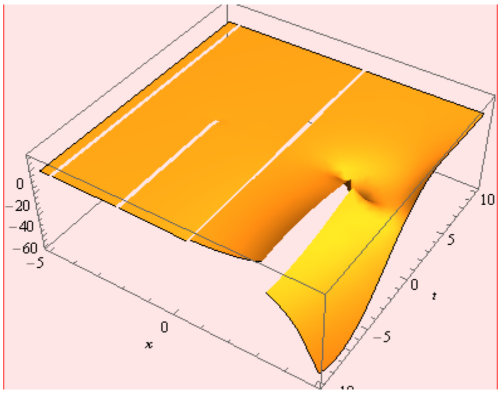
Figure 1: Graphical depiction of solutions from Eqs. (31)-(34) for $\omega = 0.15$ and $\theta = \frac{3\pi}{4}$, showing a bright soliton. Solution profile with a localized wave packet and a distinct peak.



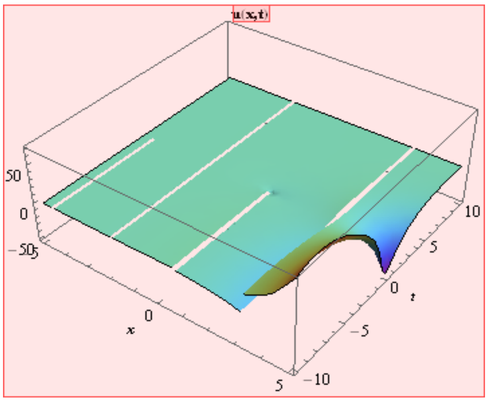
(a) (b)

(c) (d)

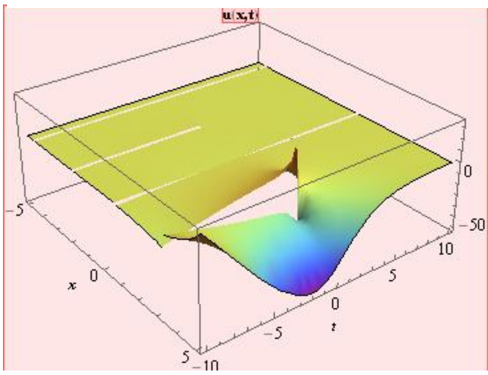
Figure 2: Graphical depiction of the solutions from eq.(42)-(45) for the values of $\omega = 0.1$, $\theta = \frac{\pi}{4}$, showing a dark Solitons, characterized by localized dips or notches



(a)



(b)



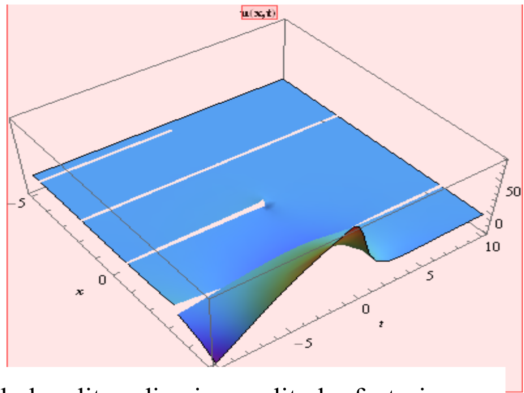
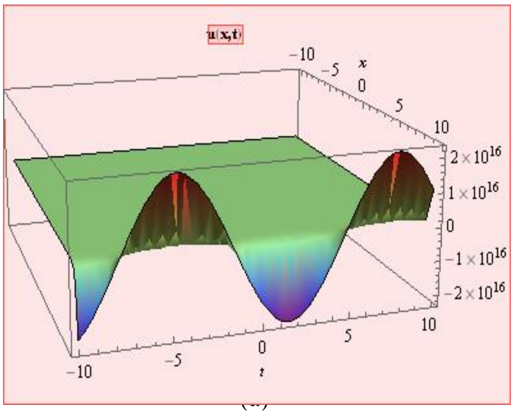
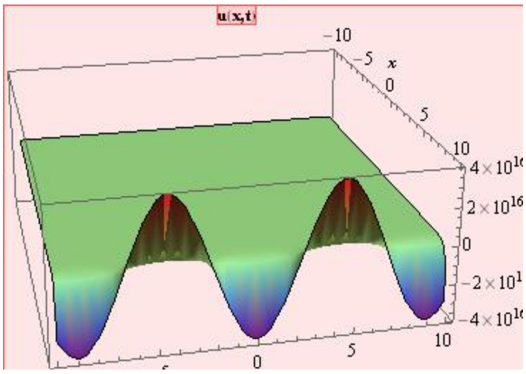


Figure 3: Graphical depiction of the model showing dark soliton dips in amplitude, featuring localized dips.



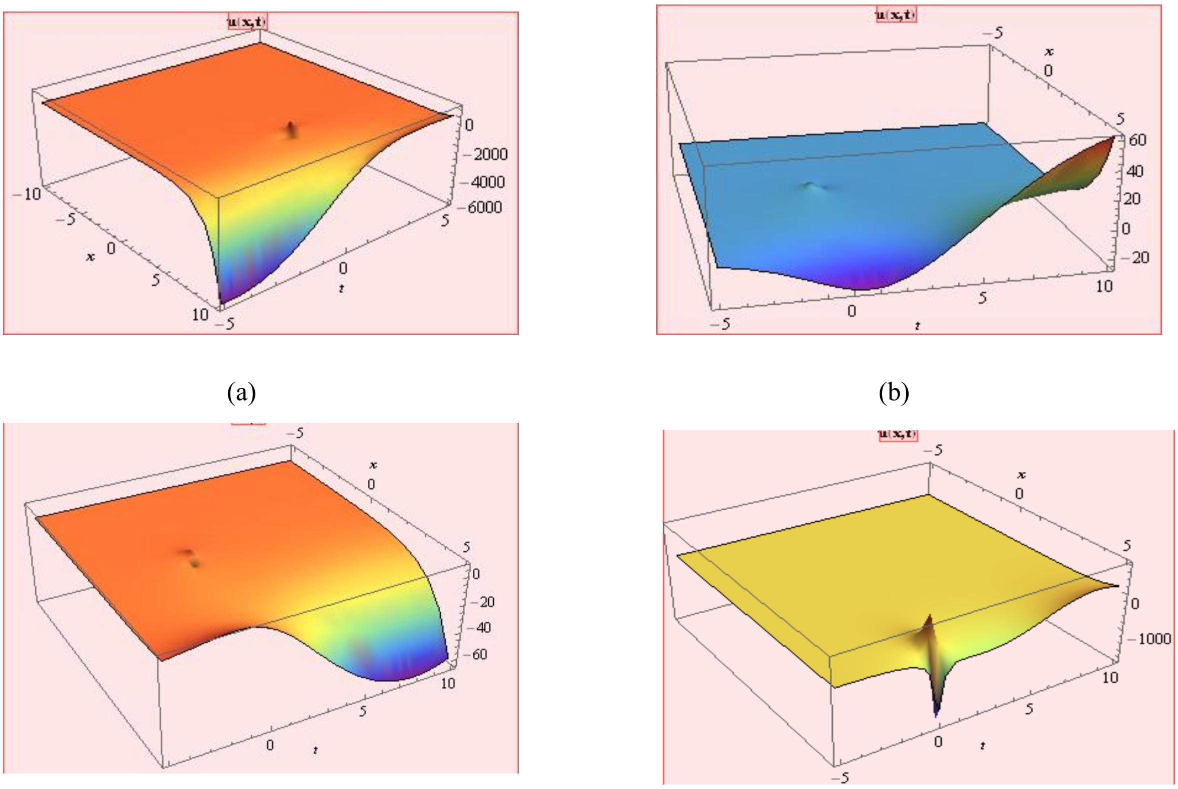
(a)



(b)

Figure 4: Graphical depiction of the solutions from eq.(67)-(68) for the values of ω = 0.5, $\theta = \frac{\pi}{4}$ showing periodic. Solitons or cnoidal wave patterns with localized peaks and troughs

.



(a) (b)

(c) (d)

Figure 5: Graphical depiction of the solutions from eq. (77)-(80) for the values of ω = 0.2, λ = 5, θ = π showing dark soliton dips in amplitude, featuring localized dips

4. CONCLUSION AND LIMITATION

This study yielded bright soliton solutions for both Kerr and QC nonlinearities. Amplitude and phase velocity were found to be sensitive to the strength of the nonlinear terms. Conservation laws confirmed that energy and momentum are preserved across all cases. Kerr nonlinearities support steeper soliton profiles, making them suitable for applications requiring high signal localization, while quadratic-cubic terms stabilize broader solitons, offering resilience against dispersion. In future work, we aim to extend the analysis to coupled NLSE systems and include higher-order perturbative effects, such as self-steepening and Raman scattering.

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