Vokasi Unesa Bulletin of Engineering, Technology and Applied Science (VUBETA)
https://journal.unesa.ac.id/index.php/vubeta
Vol. 1, No. 3, 2024, pp. 41~57 DOI: 10.26740/vubeta.v1i3.35831
ISSN: 3064-0768

Exact Solitonic Solutions in New Hamiltonian Amplitude Equation using Riccati-Bernoulli Method

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Article Info

Article history:

Received September 05, 2024
Revised October 01, 2024
Accepted November 15, 2024

Keywords:
New Hamiltonian Amplitude Equation
Kerr law
Quadratic Cubic
Parabolic law

ABSTRACT

The propagation of optical pulses in nonlinear media is a complex phenomenon that requires accurate modeling and analysis. The New Hamiltonian Amplitude Equation (HNLS) is a fundamental model that describes this phenomenon, but solving it exactly is a challenging task. We employ the Riccati-Bernoulli Sub ODE method to derive exact soliton solutions to the HNLS. This research contributes to the understanding of optical soliton dynamics in various nonlinear regimes, providing a foundation for the development of novel optical communication systems and devices. We use the Riccati-Bernoulli Sub ODE method to derive exact soliton solutions to the HNLS. The method is applied to various nonlinear regimes, including Kerr law, Quadratic Cubic, and Parabolic law nonlinearities. Additionally, we obtain particular solutions using the power series method. The resulting optical soliton solutions are expressed in terms of various mathematical functions, including trigonometric functions, hyperbolic functions, exponential functions, and rational functions. These solutions describe the oscillatory behavior, exponential growth or decay, rapid growth or decay, and algebraic decay or growth of optical pulses in various nonlinear regimes. The solutions obtained using the power series method provide further insight into the behavior of optical pulses in these regimes. Our results provide a comprehensive understanding of optical soliton dynamics in nonlinear media.

1. INTRODUCTION

The Hamiltonian Nonlinear Schrödinger Equation (HNLS) serves as a vital tool for comprehending intricate wave processes, merging nonlinearity, dispersion, and quantum mechanics [1] –[6]. The inherent nonlinearity of the equation hampers exact solutions and impedes the understanding of its structure [7]–[9][21]–[23]. The need arises to develop solutions that illuminate nonlinear wave behavior, resolve symmetry ambiguities, and establish links between theoretical predictions and experimental observations [10][11][24]–[26]. Overcoming these challenges in HNLS research holds paramount importance for its practical application in optics, condensed matter physics, and quantum information. This equation finds utility across diverse systems, spanning from optical fibers to quantum fluids [5][12]–[15][27][28]. Extending the conventional Schrödinger equation, the HNLS equation captures nonlinear effects in quantum systems through its Hamiltonian structure, encapsulating intricate wave dynamics. Originally introduced by Wadati et al. in 1992, this nonlinear Schrödinger Equation is not integrable, as evidenced by its failure to satisfy the Painlevé test [2][5][28]. The model equation is given by [1][2].

$$i\psi_x + \psi_{tt} + 2\sigma |\psi|^2 \psi - \epsilon\psi_{xt} = 0 \tag{1}$$

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where $\psi = \psi(x, t)$, and x and t represent the spatial and temporal coordinates, respectively [2][29][30]. The parameter σ denotes the coefficient of nonlinearity, while the function $F(|\psi|^2)$ characterizes the nonlinear properties of the medium ψ_x, ψ_{xt}, and ψ_{tt} correspond to various types of dispersion of optical solitons [21][22]. The equation is the well-known Schrödinger Equation with the additional term $\epsilon\psi_{x,t}$ attached, which plays a significant role in prevailing the ill-posedness of unstable nonlinear Schrödinger Equation [2]

Various research approaches have been utilized to examine the model equation, such as the F-expansion method [5], Jacobi elliptic expansion function [19], G'/G expansion method [18], Ricatti bernoulli method [26], Bernoulli sub ODE method [31][32], and numerous other techniques.

The novelty of this research lies in the exploration of precise soliton solutions of the New Hamiltonian Amplitude Equation using the Riccati-Bernoulli Sub-ODE method. This method provides accurate and reliable solutions to the nonlinear Schrödinger equation, offering valuable physical insight into the behavior of optical pulses in nonlinear media. This insight is crucial for the development of novel optical communication systems and devices.

The Riccati-Bernoulli Sub-ODE method enables the study of optical soliton dynamics, including soliton for- mation, propagation, and interaction. It also facilitates the examination of nonlinear effects, such as self-phase modulation, cross-phase modulation, and four-wave mixing, which are essential for understanding the behavior of optical pulses in nonlinear media.

A key novelty of this research is the derivation of soliton solutions, which provide valuable insight into the behavior of nonlinear optical phenomena. These solutions can be used to design and develop novel optical com- munication systems with improved performance, capacity, and reliability.

Additionally, this research utilizes the power series method to obtain accurate particular solutions to the nonlinear Schrödinger equation. This method converges to exact solutions, enabling us to obtain precise results. The power series method is a valuable tool for understanding complex nonlinear phenomena and can be used to develop novel mathematical models that describe these phenomena.

This paper is structured to provide a comprehensive overview of the proposed methodology. Section 2 delves into the intricacies of the proposed Method, offering a detailed explanation. Building on this foundation, Section 3 explores the application of the Ricatti-Bernoulli sub-ODE method to the New Hamiltonian Amplitude Equation, incorporating various nonlinearities.

The subsequent section, Section 4, employs the power series method to derive a particular solution. The results obtained are then scrutinized in Section 5, where a discussion of the findings is accompanied by physical visualizations. Finally, Section 6 presents a concluding summary of the research, acknowledging limitations and outlining potential avenues for future investigation.

2. Description of the proposed Methods

2.1 Ricatti-Bernoulli Sub ODE Methods (RBSODE)

Let Consider a given PDE [26] :

$$P(\psi, \psi_t, \psi_x, \psi_{tt}, \psi_{xx}, \psi_{tx}, \ldots) = 0. \tag{2}$$

where $\psi(x, t) = \psi(\xi)$

Stage 1:

Using the conversion

$$\psi(x, t) = \psi(\xi) \times e^{i\phi(x,t)}, \tag{3}$$

where $\xi = \lambda(x \pm vt)$ and $\phi(x, t) = -k_1 x + \omega t + \theta$. Equation (2) can be change into the accompanying ODE

$$P(\psi, \psi', \psi'', \ldots) = 0. = 0. \tag{4}$$

with $\psi(\xi) = \dfrac{\partial \psi}{\partial \xi}$

Stage 2:
Presuming that the solution to Equation (4) satisfies the Riccati-Bernoulli Equation.

$$\xi\psi' = b\psi + a\psi^{2-r} + c\psi^{r}. \tag{5}$$

with constant a,b,c and r. Taking the derivative of Equation (5). we have.

$$\psi'' = \psi^{-1-2r}(a\psi^2 + c\psi^{2-r} + c\psi^{1+r})(-a)(-2+r))\psi^2 + cr\psi^{2r} + b\psi^{1+r}. \tag{6}$$

$$\psi''' = \psi^{-2(1+r)}(b\psi + a\psi^{2-r} + c\psi^{r})(a^2(-2+r)(-3+2r))\psi^4 + c^2r(-1+2r)$$

$$\psi^{4r} + ab(-3+r)(-2+r)\psi^{3+r} + (b^2+2ac)\psi^{2+2r} + bcr(1+r\psi^{1+3r}), \ldots \tag{7}$$

Remarks
Equation (5) is Riccati equation if $ac \neq 0$ and $r = 0$. Equation (5) is Bernoulli equation

if $a \neq 0$, $c = 0$ and $r \neq 1$. To avoid the introducing new terminologies, we called Equation (5) Riccati-Bernoulli equation. Equation (5) possesses the subsequent solutions.:

Classification of solution:

Case 1: If $r = 1$, Equation (5) possesses the solution.

$$\psi(\xi) = Ce(b+a+c)\xi. \tag{8}$$

Case 2: If $r \neq 1$, $b = 0$, and $c = 0$, Equation (5) possesses the solution.

$$\psi(\xi) = (a(r-1)(\xi + c))\ r-1\ . \tag{9}$$

Case 3: If $r \neq 1$, $b \neq 0$, and $c = 0$, Equation (5) possesses the solution.

$$\psi(\xi) = \left(c_e(b(m-1)\xi) - \frac{a}{b}\right)^{\frac{1}{r-1}} \tag{10}$$

Case 4: If $r \neq 1$, $a \neq 0$ and $b^2 - 4ac < 0$, Equation (5) possesses the solution

$$\psi(\xi) = \left(-\frac{b}{2\alpha} + \left(\sqrt{\frac{4ac-b^2}{2a}}\tan\right)\left[\frac{(1-r)\sqrt{4ac-b\zeta}}{2}\middle|\ \xi + c\right]^{\frac{1}{1-r}}\right) \tag{11}$$

and

$$\psi(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac-b2}}{2a}\cot\left[\frac{(1-r)\sqrt{4ac-b^2}}{2}l\xi + c\right]\right)^{\frac{1}{1-r}} \tag{12}$$

Case 5: If r ≠1,a ≠ 0 and b2 − 4ac>0, Equation (5) possesses the solution.

$$\psi(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} tanh\left[\frac{(1-r)\sqrt{b^2-4ac}}{2}(\xi + c)\right]\right) \tag{13}$$

and

$$\psi(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} coth\left[\frac{(1-r)\sqrt{b^2-4ac}}{2}(\xi + c)\right]\right)^{\frac{1}{1-r}} \tag{14}$$

Case 6: If r ≠ 1,a ≠ 0 and b^2−4ac = 0, Equation (5) possesses the solution.

$$\psi(\xi) = \left(\frac{1}{a(r-1)(\xi+c)} - \frac{a}{b}\right)\frac{1}{1-r} \tag{15}$$

where C is constant.

Stage 3:

when we substitute ψ and its derivatives into Equation (4), we derive a system of algebraic equations. By choosing the value of r according to the steps discussed above, performing all necessary computation and substituting the value of $a, b, c,$ and other parameters into any of the cases Equation (8)-(15), that fit, the solution of the PDE (2) may be obtain.

3. Application of RBSODE Method to New Hamiltonian Amplitude Equation.

3.1 Kerr Law non-linearity.

Regarding the Kerr law non-linearity [32], $F(\psi) = \psi$.

$$i\psi_x + \psi_{tt} + 2\sigma|\psi|^2\psi - \epsilon\psi_{xt} = 0. \tag{16}$$

By employing Equation (3) in Equation (16) and segregating the real and imaginary components of the equation, we arrive at the following.

The imaginary component as:

$$v = \frac{\omega\epsilon-1}{k\epsilon+2\omega} \tag{17}$$

The real component as:

$$k\omega\epsilon\psi - k\psi - v^2\psi'' + v\epsilon\psi'' - 2\sigma\psi^3 + \omega^2\psi = 0 \tag{18}$$

$$\psi^0 : bcv(\epsilon - v) = 0. \tag{19}$$

$$\psi^1 : (-2acv^2 + 2acv\epsilon + b^2v(\epsilon - v) + k(\omega\epsilon - 1) + \omega^2) = 0. \tag{20}$$

$$\psi^2 : 3abv(\epsilon - v) = 0. \tag{21}$$

$$\psi^3 : -2(a\ \ v(v - \epsilon) + \sigma) = 0. \tag{22}$$

From Solving Equation (19) - (22) we obtained the following values.

$$b = 0;\ a = \frac{\sqrt{\sigma}}{\sqrt{v\epsilon - v^2}};\ \upsilon = \frac{\omega\epsilon-1}{k\epsilon+2\omega};\ \xi = t\,(\upsilon + x);\ c = \frac{k\omega\epsilon-k+\omega^2}{2av(v-\epsilon)}$$

The solution of the obtained values is given as follows.

Case A:
If $\omega, \epsilon < 0$, We acquire the subsequent solutions in terms of trigonometric functions

$$\psi 1,1 = cot\left(\frac{\frac{\sqrt{-\frac{(k\epsilon+2w)^2(k(\omega\epsilon-1)+\omega^2}{\left(\frac{1}{\omega\epsilon-1}\right)(k\epsilon^2+\omega\epsilon+1)}}\left(c+\frac{t(\omega\epsilon-1)}{k\epsilon+2\omega}+x\right)}{\sqrt{2}}}{\sqrt{2}\sqrt{\sigma}}\right)$$

$$\times\sqrt{\frac{(w\epsilon-1)(k\epsilon^2+\omega\epsilon+1)}{(k\epsilon+2\omega)^2}}\sqrt{-\frac{(kE+2\omega)^2(k(\omega\epsilon-1)+\omega^2)}{(\omega\epsilon-1)(k\epsilon2+\omega_\epsilon+1)}}\,e^i(\theta-kx+t\omega) \quad (23)$$

and

$$\psi 1,2(x,t) = tan\left(\frac{\frac{\sqrt{-\frac{(k\epsilon+2w)^2(k(\omega\epsilon-1)+\omega^2}{\left(\frac{1}{\omega\epsilon-1}\right)(k\epsilon^2+\omega\epsilon+1)}}\left(c+\frac{t(\omega\epsilon-1)}{k\epsilon+2\omega}+x\right)}{\sqrt{2}}}{\sqrt{2}\sqrt{\sigma}}\right)$$

$$\times\sqrt{\frac{(w\epsilon-1)(k\epsilon^2+\omega\epsilon+1)}{(k\epsilon+2\omega)^2}}\sqrt{-\frac{(kE+2\omega)^2(k(\omega\epsilon-1)+\omega^2)}{(\omega\epsilon-1)(k\epsilon2+\omega_\epsilon+1)}}\,e^i(\theta-kx+t\omega) \quad (24)$$

Case B:
If $\omega, \epsilon > 0$, We acquire the subsequent solutions in terms of hyperbolic functions

$$\psi 1,3(x,t) = coth\left(\frac{\frac{\sqrt{-\frac{(k\epsilon+2w)^2(k(\omega\epsilon-1)+\omega^2}{\left(\frac{1}{\omega\epsilon-1}\right)(k\epsilon^2+\omega\epsilon+1)}}\left(c+\frac{t(\omega\epsilon-1)}{k\epsilon+2\omega}+x\right)}{\sqrt{2}}}{\sqrt{2}\sqrt{\sigma}}\right)$$

$$\times\sqrt{\frac{(w\epsilon-1)(k\epsilon^2+\omega\epsilon+1)}{(k\epsilon+2\omega)^2}}\sqrt{-\frac{(kE+2\omega)^2(k(\omega\epsilon-1)+\omega^2)}{(\omega\epsilon-1)(k\epsilon2+\omega_\epsilon+1)}}\,e^i(\theta-kx+t\omega) \quad (25)$$

and

$$\psi 1,4(x,t) = tanh\left(\frac{\frac{\sqrt{-\frac{(k\epsilon+2w)^2(k(\omega\epsilon-1)+\omega^2}{\left(\frac{1}{\omega\epsilon-1}\right)(k\epsilon^2+\omega\epsilon+1)}}\left(c+\frac{t(\omega\epsilon-1)}{k\epsilon+2\omega}+x\right)}{\sqrt{2}}}{\sqrt{2}\sqrt{\sigma}}\right)$$

$$\times\sqrt{\frac{(w\epsilon-1)(k\epsilon^2+\omega\epsilon+1)}{(k\epsilon+2\omega)^2}}\sqrt{-\frac{(kE+2\omega)^2(k(\omega\epsilon-1)+\omega^2)}{(\omega\epsilon-1)(k\epsilon2+\omega_\epsilon+1)}}\,e^i(\theta-kx+t\omega) \quad (26)$$

3.2 Quadratic Cubic law non-linearity

Regarding the Quadratic Cubic law non-linearity [32], $F(\psi) = \sigma_1\sqrt{\psi} + \sigma_2\psi$ where σ_1 and σ_2 are constants

Iψx + ψtt + 2(σ1|ψ| + σ2|ψ|2) ψ−ϵψxt=0. (27)

By employing Equation (3) in Equation (27) and segregating the real and imaginary components of the equation, we arrive at the following. The imaginary component as:

$$v = \frac{\omega\epsilon - 1}{k\epsilon + 2\omega} \tag{28}$$

The real component as:

$$-k\omega\epsilon\psi + k\psi + v^2\psi'' - v\epsilon\psi'' + 2\sigma_2\psi^3 + 2\sigma_1\psi^2 - \omega^2\psi = 0 \tag{29}$$

Substituting Equation (5) and its derivatives in Equation (29) and setting $m = 0$, we obtain an overdetermine equation. By gathering terms with the same exponent of ψ^i and setting them equal to zero, we establish the subsequent system of algebraic equations.

$$\psi^0 : bcv(v-\epsilon) = 0. \tag{30}$$

$$\psi^1 : (2acv^2 - 2acv\epsilon + b^2v(v-\epsilon) - k\omega\epsilon + k - \omega^2) = 0. \tag{31}$$

$$\psi 2 : 3abv(v-\epsilon) + 2\sigma_1 = 0. \tag{32}$$

$$\psi^3 : 2(a^2v(v-\epsilon) + \sigma_2) = 0. \tag{33}$$

From Solving Equation (30) - (33) we obtained the following values:

Set 1 :

When c = 0; $a = \frac{\sqrt{\sigma_2}}{\sqrt{v\epsilon - v^2}}$; $v = \frac{\omega\epsilon - 1}{k\epsilon + 2\omega}$; $k = \frac{-9\sigma_2\omega^2 - 4\sigma_1^2}{9\sigma_2(\omega\epsilon - 1)}$; $\xi = (tv + x)$; $b = -\frac{2\sigma_1}{3av(v-\epsilon)}$

The solution of the obtained values is given as follows.

Case A :

If $\omega, \epsilon, \sigma < 0$, We acquire the subsequent solutions in terms of trigonometric functions.

$$\psi 2,1 = -\frac{1}{3\sigma_2} cot\left(\frac{1}{9}\sqrt{\frac{\left(4\sigma_1^3\epsilon - 9\sigma_1\sigma_2\omega(\omega\epsilon - 2)\right)^2}{\sigma_2^2(9\sigma_2 + 4\sigma_1^2E^2)(\omega\epsilon - 1)^2}}\left(C + \frac{902^t(w\epsilon - 1)^2}{9\sigma_2\omega(\omega t - 2) - 4\sigma_1^2 E}\right) + \sigma_1\right) \times$$
$$\left(\sqrt{o_2}\sqrt{\frac{\left(4\sigma_1^3E - 9\sigma_1\sigma_2\omega(w\epsilon - 2)\right)^2}{\sigma_2^2(9\sigma_2 + 4\sigma_1^2\epsilon^2)(\omega E - 1)^2}}\sqrt{-\frac{\sigma 2(9\sigma_2 + 4\sigma_1^2 \epsilon 2)(wt - 1)^2}{\left(4\sigma_1^2\epsilon - 9\sigma_2\omega(\omega E - 2)\right)^2}}\right) \times\ exp\left(i\left(\theta + t\omega - \frac{x(-9\sigma_2\omega^2 - 4\sigma_1^2)}{9\sigma_2(\omega E - 1)}\right)\right) \tag{34}$$

and

$$\psi 2,2 = -\frac{1}{3\sigma_2} tan\left(\frac{1}{9}\sqrt{\frac{\left(4\sigma_1^3\epsilon - 9\sigma_1\sigma_2\omega(\omega\epsilon - 2)\right)^2}{\sigma_2^2(9\sigma_2 + 4\sigma_1^2E^2)(\omega\epsilon - 1)^2}}\left(C + \frac{902^t(w\epsilon - 1)^2}{9\sigma_2\omega(\omega t - 2) - 4\sigma_1^2 E}\right) + \sigma_1\right) \times$$
$$\left(\sqrt{o_2}\sqrt{\frac{\left(4\sigma_1^3E - 9\sigma_1\sigma_2\omega(w\epsilon - 2)\right)^2}{\sigma_2^2(9\sigma_2 + 4\sigma_1^2\epsilon^2)(\omega E - 1)^2}}\sqrt{-\frac{\sigma 2(9\sigma_2 + 4\sigma_1^2 \epsilon 2)(wt - 1)^2}{\left(4\sigma_1^2\epsilon - 9\sigma_2\omega(\omega E - 2)\right)^2}}\right) \times\ exp\left(i\left(\theta + t\omega - \frac{x(-9\sigma_2\omega^2 - 4\sigma_1^2)}{9\sigma_2(\omega E - 1)}\right)\right) \tag{35}$$

Case B:

If $\omega, \epsilon, \sigma > 0$, We acquire the subsequent solutions in terms of hyperbolic functions

$$\psi 2,3 = -\frac{1}{3\sigma_2} coth\left(\frac{1}{9}\sqrt{\frac{\left(4\sigma_1^3\epsilon - 9\sigma_1\sigma_2\omega(\omega\epsilon - 2)\right)^2}{\sigma_2^2(9\sigma_2 + 4\sigma_1^2E^2)(\omega\epsilon - 1)^2}}\left(C + \frac{902^t(w\epsilon - 1)^2}{9\sigma_2\omega(\omega t - 2) - 4\sigma_1^2 E}\right) + \sigma_1\right) \times$$

$$\left(\sqrt{\sigma_2}\sqrt{\frac{\left(4\sigma_1^3E-9\sigma_1\sigma_2\omega(w\epsilon-2)\right)^2}{\sigma_2^2\left(9\sigma_2+4\sigma_1^2\epsilon^2\right)(\omega E-1)^2}}\sqrt{-\frac{\sigma2\left(9\sigma_2+4\sigma_1^2\epsilon2\right)(wt-1)^2}{\left(4\sigma_1^2\epsilon-9\sigma_2\omega(\omega E-2)\right)^2}}\right)\times exp\left(i\left(\theta+t\omega-\frac{x\left(-9\sigma_2\omega^2-4\sigma_1^2\right)}{9\sigma_2(\omega E-1)}\right)\right) \quad (36)$$

$$\psi2,4=-\frac{1}{3\sigma_2}tanh\left(\frac{1}{9}\sqrt{\frac{\left(4\sigma_1^3\epsilon-9\sigma_1\sigma_2\omega(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(9\sigma_2+4\sigma_1^2E^2\right)(\omega\epsilon-1)^2}}\left(C+\frac{902^t(w\epsilon-1)^2}{9\sigma_2\omega(\omega t-2)-4\sigma_1^2E}\right)+\sigma_1\right)\times$$
$$\left(\sqrt{\sigma_2}\sqrt{\frac{\left(4\sigma_1^3E-9\sigma_1\sigma_2\omega(w\epsilon-2)\right)^2}{\sigma_2^2\left(9\sigma_2+4\sigma_1^2\epsilon^2\right)(\omega E-1)^2}}\sqrt{-\frac{\sigma2\left(9\sigma_2+4\sigma_1^2\epsilon2\right)(w\epsilon-1)^2}{\left(4\sigma_1^2\epsilon-9\sigma_2\omega(\omega\epsilon-2)\right)^2}}\right)\times exp\left(i\left(\theta+t\omega-\frac{x\left(-9\sigma_2\omega^2-4\sigma_1^2\right)}{9\sigma_2(\omega E-1)}\right)\right) \quad (37)$$

3.3 Parabolic law non-linearity

Regarding the Parabolic law non-linearity [32], $F(\psi)=\sigma_1\psi+\sigma_2\psi^2$. where σ_1 and σ_2 are constants.

$$i\psi_x+\psi_{tt}+2(\sigma_1|\psi|^2+\sigma_2|\psi|^4)\psi-\epsilon\psi_{xt}=0 \quad (38)$$

By employing Equation (3) in Equation (43) and segregating the real and imaginary components of the equation, we arrive at the following.
The imaginary component as:

$$v=\frac{\omega\epsilon-1}{k\epsilon+2\omega} \quad (39)$$

The real component as:

$$-k\omega\epsilon\psi+k\psi+v^2\psi''-v\epsilon\psi''+2\sigma_2\psi^5+2\sigma_1\psi^3-\omega^2\psi=0 \quad (40)$$

Setting

$$\psi=v^{\frac{1}{2}} \quad (41)$$

Equation (58) is transfom to

$$-4V^2(k(\omega\epsilon-1)+\omega^2)+2v(v-\epsilon)VV''+v(\epsilon-v)V'^2+8\sigma_2V^4+8\sigma_1V^3=0 \quad (42)$$

Substituting Equation (5) and its derivatives in Equation (42) and setting $m=0$, we obtain an overdeterming equation. By gathering terms with the same exponent of ψ^i and setting them equal to zero, we establish the subsequent system of algebraic equations.

$$V^0: c^2v(\epsilon-v)=0. \quad (43)$$

$$V^2: (2acv^2-2acv\epsilon+b^2v(v-\epsilon)+k(4-4\omega\epsilon)-4\omega^2)=0. \quad (44)$$

$$V^3: (4abv(v-\epsilon)+8\sigma_1)+c^2v(\epsilon-v)=0. \quad (45)$$
$$V^4: (3a^2v(v-\epsilon)+8\sigma_2)=0. \quad (46)$$

From Solving Equation (43) - (46), we obtained the following values:

$$c=0\,;\,a=\frac{2\sqrt{\frac{2}{3}}\sqrt{\sigma2}}{\sqrt{v\epsilon-v^2}}\,;\,v=\frac{w\epsilon-1}{k\epsilon+2\omega}\,;\;k=\frac{-8\sigma_2\omega^2-3\sigma_1^2}{8\sigma_2(\omega\epsilon-1)}\,;\,\xi=tv+x\,;\,b=-\frac{2\sigma_1}{av(v-\epsilon)}$$

The solution of the obtained values is given as follows.

Case A :
If $\omega,\epsilon,\sigma<0$, We acquire the subsequent solutions in terms of trigonometric functions.

$$\psi 3{,}1 = \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{\sigma_2}\sqrt{\frac{\left(3\sigma_1^3\epsilon - 8\sigma{,}02^W(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(8\sigma_2+3\sigma_1^2\epsilon^2\right)^1(\omega\epsilon-1)^2}} \times \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{\sigma_2}\sqrt{\frac{\left(3\sigma_1^3\epsilon - 8\sigma{,}02^W(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(8\sigma_2+3\sigma_1^2\epsilon^2\right)^1(\omega\epsilon-1)^2}} \times$$
$$cot\left(\frac{1}{8}\sqrt{3}\sqrt{\frac{(3\sigma_1^3\epsilon-8\sigma_1\sigma_2\omega C\omega\epsilon-2)^2}{\sigma_2^2(8\sigma_2+3\sigma_1^2\epsilon^2)(cot-1)^2}}\left(C+\frac{8\sigma_2 t(\omega\epsilon-1)^2}{8\sigma_2\omega(\omega\epsilon-2)-3\sigma_1^2\epsilon}+x\right)+\sigma_1\times\right.$$
$$\left.exp\left(i\left(\theta+tw-\frac{x(-8\sigma_2\omega^2-3\sigma_1^2)}{8\sigma_2(w\epsilon-1)}\right)\right)\right)\frac{1}{2} \tag{47}$$

and

$$\psi 3{,}2 = \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{\sigma_2}\sqrt{\frac{\left(3\sigma_1^3\epsilon - 8\sigma{,}02^W(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(8\sigma_2+3\sigma_1^2\epsilon^2\right)^1(\omega\epsilon-1)^2}} \times \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{\sigma_2}\sqrt{\frac{\left(3\sigma_1^3\epsilon - 8\sigma{,}02^W(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(8\sigma_2+3\sigma_1^2\epsilon^2\right)^1(\omega\epsilon-1)^2}} \times$$
$$tan\left(\frac{1}{8}\sqrt{3}\sqrt{\frac{(3\sigma_1^3\epsilon-8\sigma_1\sigma_2\omega C\omega\epsilon-2)^2}{\sigma_2^2(8\sigma_2+3\sigma_1^2\epsilon^2)(cot-1)^2}}\left(C+\frac{8\sigma_2 t(\omega\epsilon-1)^2}{8\sigma_2\omega(\omega\epsilon-2)-3\sigma_1^2\epsilon}+x\right)+\sigma_1\times\right.$$
$$\left.exp\left(i\left(\theta+tw-\frac{x(-8\sigma_2\omega^2-3\sigma_1^2)}{8\sigma_2(w\epsilon-1)}\right)\right)\right)\frac{1}{2} \tag{48}$$

Case B :
If ω,ϵ,σ > 0, We acquire the subsequent solutions interms of hyperbolic functions.

$$\psi 3{,}3 = \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{\sigma_2}\sqrt{\frac{\left(3\sigma_1^3\epsilon - 8\sigma{,}02^W(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(8\sigma_2+3\sigma_1^2\epsilon^2\right)^1(\omega\epsilon-1)^2}} \times \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{\sigma_2}\sqrt{\frac{\left(3\sigma_1^3\epsilon - 8\sigma{,}02^W(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(8\sigma_2+3\sigma_1^2\epsilon^2\right)^1(\omega\epsilon-1)^2}} \times$$
$$coth\left(\frac{1}{8}\sqrt{3}\sqrt{\frac{(3\sigma_1^3\epsilon-8\sigma_1\sigma_2\omega C\omega\epsilon-2)^2}{\sigma_2^2(8\sigma_2+3\sigma_1^2\epsilon^2)(cot-1)^2}}\left(C+\frac{8\sigma_2 t(\omega\epsilon-1)^2}{8\sigma_2\omega(\omega\epsilon-2)-3\sigma_1^2\epsilon}+x\right)+\sigma_1\times\right.$$
$$\left.\exp\left(i\left(\theta+tw-\frac{x(-8\sigma_2\omega^2-3\sigma_1^2)}{8\sigma_2(w\epsilon-1)}\right)\right)\right)\frac{1}{2} \tag{49}$$

and

$$\psi 3{,}4 = \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{\sigma_2}\sqrt{\frac{\left(3\sigma_1^3\epsilon - 8\sigma{,}02^W(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(8\sigma_2+3\sigma_1^2\epsilon^2\right)^1(\omega\epsilon-1)^2}} \times \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{\sigma_2}\sqrt{\frac{\left(3\sigma_1^3\epsilon - 8\sigma{,}02^W(\omega\epsilon-2)\right)^2}{\sigma_2^2\left(8\sigma_2+3\sigma_1^2\epsilon^2\right)^1(\omega\epsilon-1)^2}} \times$$
$$tanh\left(\frac{1}{8}\sqrt{3}\sqrt{\frac{(3\sigma_1^3\epsilon-8\sigma_1\sigma_2\omega C\omega\epsilon-2)^2}{\sigma_2^2(8\sigma_2+3\sigma_1^2\epsilon^2)(cot-1)^2}}\left(C+\frac{8\sigma_2 t(\omega\epsilon-1)^2}{8\sigma_2\omega(\omega\epsilon-2)-3\sigma_1^2\epsilon}+x\right)+\sigma_1\times\right.$$
$$\left.\exp\left(i\left(\theta+tw-\frac{x(-8\sigma_2\omega^2-3\sigma_1^2)}{8\sigma_2(w\epsilon-1)}\right)\right)\right)\frac{1}{2} \tag{50}$$

4. Particular solution of Hamiltonian Amplitude NLSE

To find the particular solution of Hamiltonian Amplitude NLSE we need to convert the equation into system of equations we using the transformation [1][26][40]

$$\psi = u + iv \tag{51}$$

where u and v are real-valued functions.
substituting Equation (51) into Equation (1) we have

$$-v_x + u_{tt} - \epsilon u_{xt} + 2\sigma(u^2 + v^2)u = 0$$

$$u_x + v_{tt} - \epsilon u_{xt} + 2\sigma(u^2 + v^2)v = 0 \tag{52}$$

Conversion of system of PDEs to ODEs

Using the transformation $\xi = x + ct$ and express u and v as functions of ξ:

$u(\xi) = u(x + ct)$

$v(\xi) = v(x + ct)$

Using the chain rule, we compute the necessary derivatives:

$$\frac{\partial}{\partial x} = \frac{d\xi}{dx}\frac{d}{d\xi} = 1 \cdot \frac{d}{d\xi} = \frac{d}{d\xi}$$

$$\frac{\partial}{\partial t} = \frac{d\xi}{dt}\frac{d}{d\Sigma} = c \cdot \frac{d}{d\xi} = c\frac{d}{d\xi}$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial t}\left(C\frac{d}{d\xi}\right) = C\frac{d}{d\xi}\left(C\frac{d}{d}\xi\right) = C^2\frac{d^2}{d\xi^2}$$

$$\frac{\partial^2}{\partial x\partial t} = \frac{\partial}{\partial x}\left(C\frac{d}{d\xi}\right) = C\frac{d^2}{d^2\xi^2}$$

Substituting the derivative into the Pdes equations we get:

$$-v' + (c^2 - \epsilon c)u'' + 2\sigma(u^2 + v^2)u = 0$$
$$u' + (c^2 - \epsilon c)v'' + 2\sigma(u^2 + v^2)v = 0 \quad (53)$$

To solve the system of ordinary differential equations (ODEs) using power series solutions, we express $u(\xi)$ and $v(\xi)$ as [26]:

$$u(\xi) = \sum_{n=0}^{\infty} a_n\xi^n \text{ , } u'(\xi) = \sum_{n=1}^{\infty} a_n n\,\xi^{n-1}, u''(\xi) = \sum_{n=2}^{\infty} a_n n(n-1)\xi^{n-2} \quad (54)$$

$$v(\xi) = \sum_{n=0}^{\infty} b_n\xi^n, v'(\xi) = \sum_{n=1}^{\infty} b_n n\,\xi^{n-1}, v''(\xi) = \sum_{n=2}^{\infty} b_n n(n-1)\xi^{n-2} \quad (55)$$

where a_n and b_n are the coefficients to be determined.
Substitute the power series and their derivatives into the ODEs and simplify we have:

$$-\sum_{m=0}^{\infty} b_{m+1}(n+1)\xi^m + (c^2 - \epsilon c\sum_{m=o}^{\infty} a_{m+2}(m+1)\xi^m + 2\sigma)\ [(\sum_{m=0}^{\infty} a_m\xi^m)^2 + (\sum_{m=0}^{\infty} b_m\xi^n)^2](\sum_{m=0} a_m\xi^m) = 0$$

$$\sum_{m=0}^{\infty} a_{m+1}(n+1)\xi^m + (c^2 - \epsilon c\sum_{m=o}^{\infty} b_{m+2}(m+1)\xi^m + 2\sigma)\ [(\sum_{m=0}^{\infty} a_m\xi^m)^2 + (\sum_{m=0}^{\infty} b_m\xi^n)^2](\sum_{m=0} b_m\xi^m) = 0$$

For m = 0 we have

$$a_2 = \frac{b_1 - 2\sigma[a_0^3 + a_0 + b_0^2]}{2(c^2 - \epsilon c)}, \qquad b_2 = \frac{a_1 - 2\sigma[b_0^3 + a_0^2 b_0]}{2(c^2 - \epsilon c)} \quad (56)$$

After expanding and simplifying we get

$$-b_{m+1}(m+1) + (c^2 - c)a_{m+2}(m+2)(m+1) + 2\sigma\left(\left(\sum_{1=0}^{m}\left(\sum_{j:0}^{1} a_j a_l - j\right)a_{m_{-1}}\right) + \left(\sum_{j=0}^{m}\left(\sum_{j}^{1} b_j b_l - j\right)a_{m_{-l}}\right)\right) \quad (57)$$

$$a_{m+1}(m+1)+(c^2-c)b_{m+2}(m+2)(m+1)+2\sigma\left(\left(\sum_{1=0}^{m}\left(\sum_{j:0}^{1}a_j a_l - j\right)b_{m_{-1}}\right)+\left(\sum_{j=0}^{m}\left(\sum_{j}^{1}b_j b_l - j\right)b_{m_{-l}}\right)\right) \quad (58)$$

Generally, for m ≥ 0 we can have

$$a_{m+2}=\frac{b_{m+1}}{(m+2)(c^2-\epsilon c)}+\frac{2\sigma}{(m+1)(m+2)(c^2-\epsilon c)}\left(\sum_{l=0}^{m}\sum_{j=0}^{l}a_j a_{l^-j} a_{m_{-l}}\right)+\left(\sum_{l=0}^{m}\sum_{j=0}^{l}b_j b_{l^-j} a_{m_{-l}}\right) \quad (59)$$

$$b_{m+2}=\frac{\alpha_{m+1}}{(m+2)(c^2-\epsilon c)}+\frac{2\sigma}{(m+1)(m+2)(c^2-\epsilon c)}\left(\sum_{l=0}^{m}\sum_{j=0}^{l}a_j a_{l^-j} b_{m_{-l}}\right)+\left(\sum_{l=0}^{m}\sum_{j=0}^{l}b_j b_{l^-j} b_{m_{-l}}\right) \quad (60)$$

For m =1:

$$a_3=\frac{b_2}{3(c^2-\epsilon c)}+\frac{2\sigma}{6(c^2-\epsilon c)}(3a_0^2a+b_0^2a,+2b_0b_1a_0) \quad (61)$$

$$b_3=\frac{a_2}{3(c^2-\epsilon c)}+\frac{2\sigma}{6(c^2-\epsilon c)}(a_0^2b,+2a_0a_1b,+2b_0^2b) \quad (62)$$

For m = 2:

$$a_4=\frac{b_3}{4(c^2-\epsilon c)}+\frac{2\sigma}{12(c^2-\epsilon c)}(3a_0^2a_2^{+2}a_Oa_1^2+a_1^2a_0+3b_0^2a_2+2b_0b_1a_1+2b_0b_2a_0+b^2a_0) \quad (63)$$

$$b_4=\frac{a_3}{4(c^2-\epsilon c)}+\frac{2\sigma}{12(c^2-\epsilon^c)}$$

$$(a_0^2b^2+2a_0a_1b_1+2a_0a_2b_0+a^2,b_0rb_0^2b_2+2b_0b_1^2+2b_0b_2b_0+b_1^2b_0) \quad (64)$$

Then the power series solution of the equation can be written as

$$F(\xi)=a_0(\xi)+a_1(\xi)+a_2(\xi)+a_3(\xi)+a_{m+2}(\xi) \quad (65)$$

$$G(\xi)=b_0(\xi)+b_1(\xi)+b_2(\xi)+b_3(\xi)+b_{m+2}(\xi) \quad (66)$$

$$F(\xi)=a_0(\xi)+a_1(\xi)+\left(\frac{b_1-2\sigma(a_0^3+a_0+b_0^2)}{2(c^2-\epsilon c)}\right)\xi^2+\left(\frac{b2}{3(c^2-\epsilon c)}+\frac{2\sigma}{6(c^2-\epsilon c)}(3a_0^2a_1+b_0^2a_1+2b_0b_1a_0)\right)\xi^3+\left(\frac{b_2}{4(c^2-\epsilon c)}+\frac{2\sigma}{12(c^2-\epsilon c)}(3a_0^za_2+2a_0a_1^2+a_1^2a_0+3b_0^2a_2+2b_0b_1a,+2b_0b_2\alpha_0+b_1^2\alpha_0)\right)\xi^4+\left(\frac{b_{m+1}}{(m+2)(c^{2-}\in c)}+\frac{2\sigma}{(m+1)(m+2)}\left(\sum_{1=0}^{m}\sum_{j=0}^{l}a_ja_l-j^am_{-1}+\sum_{1=0}^{m}\sum_{j=0}^{l}b_jb_l-ja_{m-1}\right)\right)\xi^{m+2} \quad (67)$$

$$G(\xi) = \alpha_0(\xi) + b_1(\xi) - \left(\frac{\alpha_1 - 2\sigma(b_0^3 + \alpha_0^2 b_0)}{2(c^2 - \epsilon c)}\right)\xi^2 + \left(\frac{b2}{3(c^2-\epsilon c)} + \frac{2\sigma}{6(c^2-\epsilon c)}(a_0^2 b_1 + 2\alpha_0\alpha_1 b_0 + b_0^2 a_1 + 2b_0^2 b_1 + 2b_0^2 b_1)\right)\xi^3 + \left(\frac{\alpha_3}{4(c^2-\epsilon c)} + \frac{2\sigma}{12(c^2-\epsilon c)}\left(a_0^2 b_2 + 2a_0\alpha_{1b_1} + 2\alpha_0\alpha_2 b_0 + a_1^2 b_0 + b_0^2 a_2 + 2b_0 b_1^2 + 2b_0 b_2 b_0 + b_1^2 b_0\right)\right)\xi^4 + \left(\frac{\alpha_{m+1}}{(m+2)(c^{2-}\epsilon c)} + \frac{2\sigma}{(m+1)(m+2)}\left(\sum_{1=0}^{m}\sum_{j=0}^{l} a_j a_l - j^b m_{-1} + \sum_{1=0}^{m}\sum_{j=0}^{l} b_j b_l - j b_{m-1}\right)\right)\xi^{m+2} \quad (68)$$

Then the power series solution of the equation is

$$\psi = \sum_{m=0}^{\infty} a_m \xi^m + i\sum_{m=0}^{\infty} b_m \xi^m$$

$$\psi = (a_0(\xi) + a_1(\xi) + \left(\frac{b_1 - 2\sigma(a_0^3 + a_0 + b_0^2)}{2(c^2-\epsilon c)}\right)\xi^2 + \left(\frac{b_2}{3(c^2-tc)} + \frac{2\sigma}{6(c^2-\epsilon c)}(3a_0^2 a_1 + b_0^2 a_1 + 2b_0 b_1 a_0)\right)\xi^3 + \left(\frac{b_3}{3(c^1-\epsilon c)} + \frac{2\sigma}{12(c^2-\epsilon c)}(3\alpha_0^2\alpha_2 + 2\alpha_0\alpha_1^2 + \alpha_1^2\alpha_0 + 3b_0^2\alpha_2 + 2b_0 b_1\alpha_1 + 2b_0 b_2\alpha_0 + b_1^2\alpha_0)\right)\xi^4 + \left(\frac{bm_{+1}}{(m+2)(c^2-Ec)} + \frac{2\sigma}{(m+1)(n+2)(c^2-\epsilon c)}\left(\sum_{1=0}^{n}\sum_{j=0}^{L} \mathrm{a}_j a, -j^a m - 1 + \sum_{l=0}^{m}\sum_{j=0}^{l} b_j b_l - j b_{m-1}\right)\xi^{m+2}\right) + i(b_0(\xi) + b_1(\xi) - \left(\frac{a_1 2\sigma[b_0^3 + a_0^2 b_0]}{2(c^2-\epsilon c)}\right)\xi^2 + \left(\frac{a_2}{3(c^2-\epsilon\varepsilon)} + \frac{2\sigma}{b(c^2-EC)}(a_0^3 b_1 + 2\alpha_0\alpha_1 b_0 + b_0^2 b_1 + 2b_0^2 b_0)\right)\xi^3 + \left(\frac{a_3}{u(c^2-\epsilon c)} + \frac{2\sigma}{12(c^2-\epsilon c)}(\alpha_0^2 b_2 + 2\alpha_0\alpha_1 b_1 + 2\alpha_0\alpha_2 b_0 + \alpha_1^2 b_0 + b_0^2 b_2 + 2b_0 b_1^2 + 2b_0 b_2 b_0 + b_1^2 b_0)\right)\xi^4 + \left(\frac{am+1}{(m+2)(c^2-\epsilon c)} + \frac{2\sigma}{(m+1)(m+2)(c^2-\epsilon c)}\left(\sum_{1=0}^{m}\sum_{j=0}^{L} a_j a_1 - jbm - 1\right) + \left(\sum_{1=0}^{m}\sum_{j=0}^{L} b_j b_1 - jbm - 1\right)\right)\xi^{m+2}) \quad (69)$$

Finally, substitute ξ = x + ct into the series.

5. RESULT AND DISCUSSION

Employing the proposed method to tackle the New Hamiltonian Amplitude Nonlinear Schr¨odinger Equation (HNLS) unlocks significant advantages in our methodology, offering a robust framework for deriving precise solutions to this complex equation. This approach shines in its capability to yield accurate solutions for the HNLS equation, particularly in specific scenarios where traditional methods may falter.

The crux of our method involves transforming the intricate HNLS equation into a more comprehensible form, thereby unveiling profound insights into the behavior of nonlinear waves across a range of systems. By doing so, we can distill the essence of the HNLS equation, revealing the underlying dynamics that govern the evolution of nonlinear waves.

One of the key benefits of our approach is its ability to capture soliton solutions, which characterize steadfast, localized waveforms that retain their structure as they propagate. These solutions provide a window into the delicate interplay between nonlinear self-interactions and dispersive effects, thereby enriching our comprehension of how nonlinearities ultimately influence the dynamics of waves.

The significance of soliton solutions cannot be overstated, as they play a critical role in understanding various phenomena in physics, including optical fibers, Bose-Einstein condensates, and ocean waves. By deriving precise soliton solutions to the HNLS equation, we can gain valuable insights into the behavior of these nonlinear systems, ultimately informing the development of novel technologies and applications.

Furthermore, our method offers a flexible framework for exploring the rich dynamics of the HNLS equation. By systematically analyzing the solutions obtained through our approach, we can uncover novel patterns and structures that may have eluded traditional methods. This, in turn, can lead to new avenues of research, as we continue to push the boundaries of our understanding of nonlinear wave dynamics.

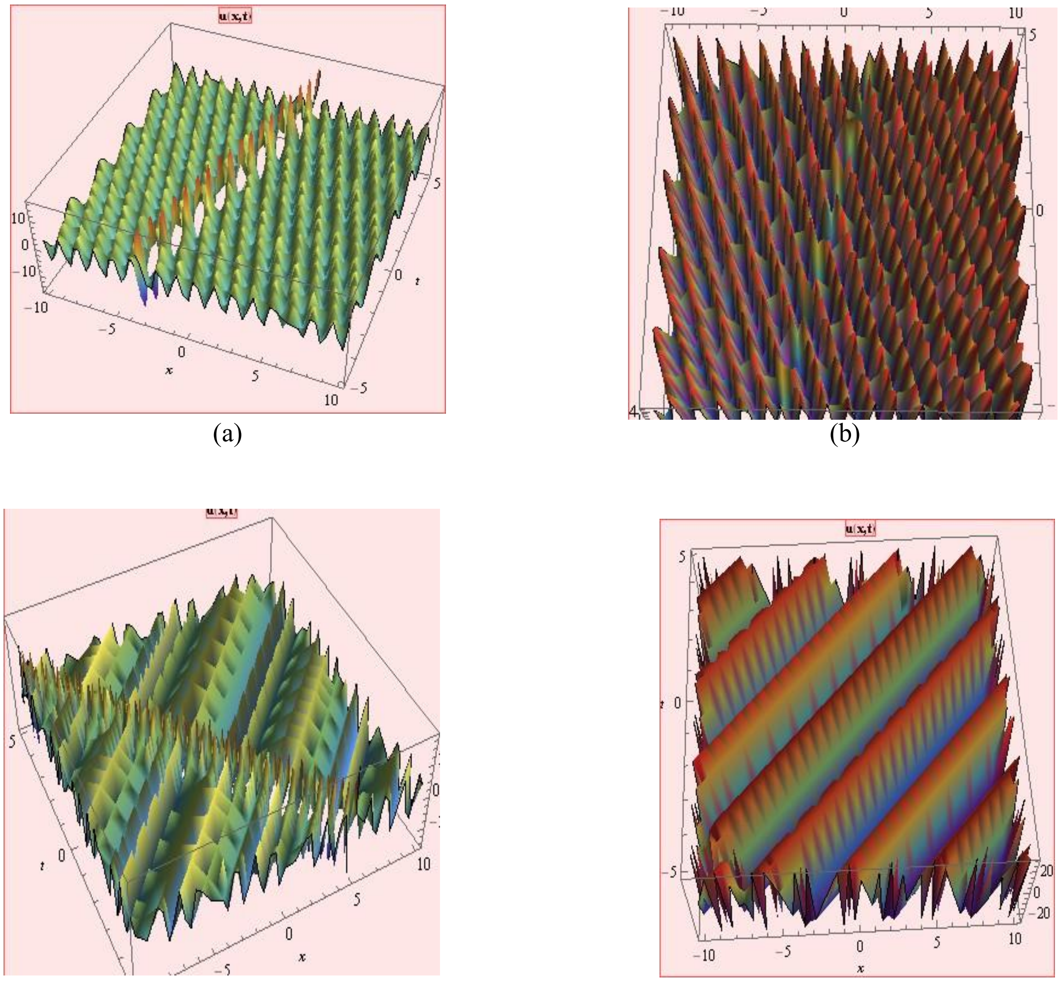
Finally, we successfully obtained the particular solution of the model by employing the power series method, which enabled us to derive an accurate and reliable solution, providing valuable insights into the behavior of the system under consideration.

Physical Visualization:

The behavior of these solutions is elucidated through 3D visualizations in the figures below.
Figure 1. below shows the behavior of kerr law non linearity for the values of $\epsilon = -5, \omega = 3, k = -5, C = 1, \sigma = 2, \theta = \pi.$

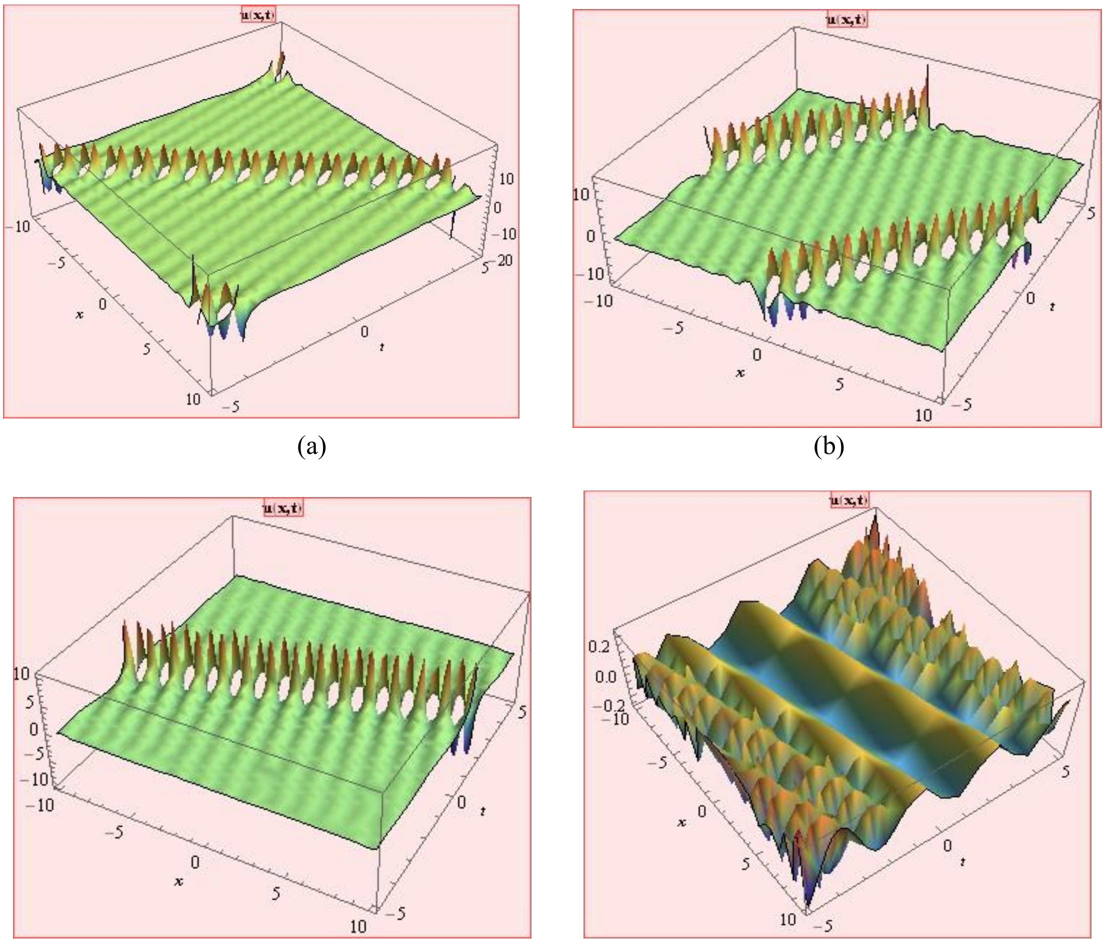
Figure 2. below shows the behavior of Quadratic law non linearity for the values of $\epsilon = 1, \omega = 2, k = 5, C = 0.5, \sigma_1 = 2, \sigma_2 = 3, \theta = \frac{\pi}{2}$

Figure 3. below shows the behavior of Parabolic law non linearity for the values of $\epsilon = 1, \omega = 5, k = 5, C = 5, \sigma = 2, \sigma_2 = 3, \theta = \pi.$



(a) (b) (c) (d)

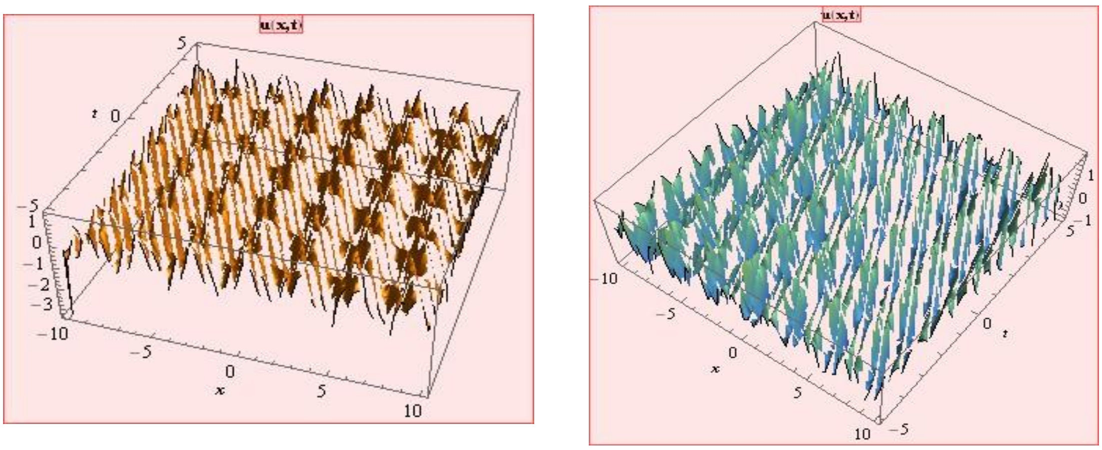
Figure 1. The sketch is the dynamic behavior of solution of kerr law non linearity for different values of free parameters in the space interval $[-10 \leq x \leq 10]$ and time interval $[-5 \leq t \leq 5]$



(a) (b)

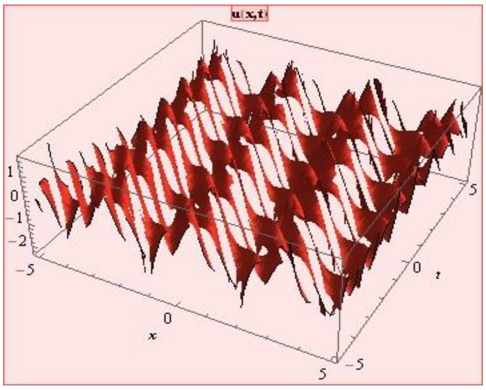
(c) (d)

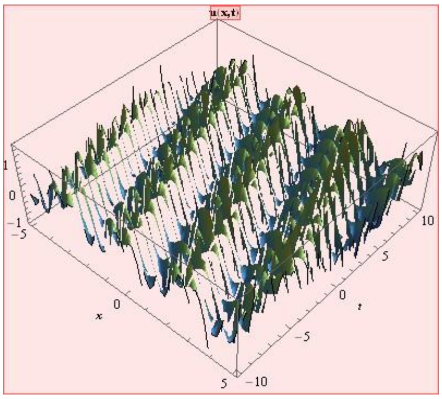
Figure 2. The sketch is the dynamic behavior of solutions of Quadratic cubic non linearity for different values of free parameters in the space interval $[-10 \leq x \leq 10]$ and time interval $[-5 \leq t \leq 5]$



(a) (b)

]





(c) (d)

Figure 3. The sketch is the dynamic behavior of solutions of Parabolic law non linearity in the space interval $[-10 \leq x \leq 10]$ and time interval $[5 \leq t \leq 5]$

6. CONCLUSIONS AND LIMITATION

In conclusion, this research has successfully employed the Riccati-Bernoulli Sub ODE method to derive exact soliton solutions to the Hamiltonian Nonlinear Schr¨odinger Equation (HNLS), thereby providing a comprehensive understanding of optical soliton dynamics in nonlinear media. The solutions obtained have shed light on the oscillatory behavior, exponential growth or decay, rapid growth or decay, and algebraic decay or growth of optical pulses in various nonlinear regimes. The findings of this research significantly contribute to the field, offering valuable insights into the intricate complexities of nonlinear wave phenomena. As we continue to navigate the complexities of nonlinear wave phenomena, the insights gleaned from this research will undoubtedly serve as a beacon, illuminating the path forward.

- The research focuses on a specific model (HNLS) and may not be generalizable to other models or nonlinear media.
- The solutions obtained are exact soliton solutions, which may not capture the full complexity of optical pulse propagation in nonlinear media.
- The research may not account for experimental imperfections or noise that can affect optical pulse propaga- tion.

Future direction of research

- Investigating the applicability of the Riccati-Bernoulli Sub ODE method to other nonlinear models or equa- tions.
- Exploring the effects of experimental imperfections or noise on optical pulse propagation in nonlinear media.
- Apply numerical methods to simulate optical pulse propagation in nonlinear media, complementing the exact solutions obtained in this research.
- Investigating the potential applications of the obtained soliton solutions in optical communication systems and devices.

ACKNOWLEDGMENT

The author extends heartfelt gratitude to the Vokasi Unesa Bulletin of Engineering, Technology, and Applied Science (VUBETA) for providing the opportunity to disseminate this research to the scientific community. The editorial team’s expert guidance, technical assistance, and publication endeavors are deeply acknowledged and greatly valued.

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