



A COMPOUND POISSON DISTRIBUTION WITH RIEMANN ZETA SUMMANDS: CHARACTERIZATION AND INFINITE DIVISIBILITY

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ABSTRACT

Infinitely divisible distributions constitute an important class of probability distributions due to their fundamental role in convolution structures, limit theorems, and Lévy processes. The Riemann zeta distribution, obtained from the Riemann zeta function through the transformation ($X = \log n$), is a discrete infinitely divisible distribution whose properties are closely related to the von Mangoldt function. Motivated by this connection, this paper investigates a compound Poisson distribution generated by a random sum of independent and identically distributed Riemann zeta random variables. The characteristic function of the resulting distribution is derived using the Fourier–Stieltjes transform. By constructing an auxiliary family of functions satisfying the necessary properties of characteristic functions, we prove the infinite divisibility of the proposed compound distribution. Furthermore, we show that the corresponding characteristic function is uniformly continuous, positive definite, and has no real zeros. These results provide a number-theoretic extension of existing characterizations of compound Poisson distributions with other infinitely divisible summands, including geometric and Cauchy-variance distributions.

Keywords: Characteristic function, Compound Poisson Distribution, Infinite Divisibility, Riemann Zeta Distribution, von Mangoldt Function

1. Introduction

Infinitely divisible distributions constitute a fundamental class of probability distributions due to their important roles in convolution structures, limit theorems, and stochastic processes with stationary independent increments, particularly Lévy processes [1, 2, 3]. A distribution F is said to be infinitely divisible if, for every positive integer n , there exist independent and identically distributed random variables $X_{n1}, \dots, X_{nn}$ such that their sum has distribution F . This concept provides a natural framework for studying distributions that can be decomposed into arbitrary numbers of independent components.

The Riemann zeta function has been extensively investigated in analytic number theory, statistics, and mathematical analysis. Euler first established its fundamental properties for real arguments, while Riemann [1] extended the definition to complex variables in 1859. Khinchine [2] first demonstrated that the characteristic function of the Riemann zeta distribution is infinitely divisible, a result later included in the work of Gnedenko and Kolmogorov [3]. This infinite divisibility property is closely related to the representation of the characteristic function through the von Mangoldt function, a classical arithmetic function defined on prime powers.

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The compound Poisson distribution, defined as the sum of a Poisson-distributed number of independent and identically distributed random variables, represents one of the most important examples of infinitely divisible distributions [4]. Owing to its flexible convolution structure, compound Poisson models have been widely applied in actuarial science, risk theory, and stochastic modeling, where their characteristic functions provide a convenient representation of aggregate random quantities [5, 6]. Various studies have investigated compound Poisson distributions with different choices of summand distributions. For example, compound Poisson approximation results have been developed for sums of dependent or non-negative random variables [7, 8], while nonparametric estimation methods have been proposed for compound Poisson processes [9]. More recently, compound Poisson models have also been used to construct frameworks for weighted network data, with applications to systemic risk assessment in finance [10]. Zhang and Li [11, 12] studied discrete compound Poisson distributions, and Yurinanda, Yanuar, and Devianto [13] established continuity and positive definiteness properties of the characteristic function for a compound Poisson distribution generated by geometric random variables. Related investigations have also considered compound Poisson models with Cauchy-variance summands. Furthermore, extensions from infinite divisibility to quasi-infinite divisibility have been explored, where a characteristic function is represented as the ratio of two infinitely divisible characteristic functions [14, 15].

These developments indicate that the choice of the summand distribution plays an important role in determining the analytical structure and properties of the resulting compound Poisson distribution. In particular, when the summands possess an established infinite divisibility structure, investigating how this structure is reflected in the corresponding compound distribution provides a natural problem in distribution theory. This motivates the study of compound Poisson models with Riemann zeta summands, for which the arithmetic structure of the underlying distribution introduces an additional connection between probability theory and number-theoretic functions.

Despite these developments, the compound Poisson distribution generated by independent and identically distributed Riemann zeta random variables has not yet been characterized. Since the Riemann zeta distribution possesses infinite divisibility and is naturally associated with the complex parameter $s = \sigma + it$, its incorporation into the compound Poisson framework leads to analytical questions beyond those encountered in previously studied cases. In this paper, we investigate the compound Poisson distribution

$$S = X_1 + X_2 + \cdots + X_N, \quad (1)$$

where N follows a Poisson distribution and $X_1, X_2, \dots$ are independent and identically distributed Riemann zeta random variables with parameter $s = \sigma + it$.

The main contributions of this paper are summarized as follows:

1. We derive the characteristic function of the compound Poisson distribution generated by Riemann zeta random variables using the Fourier–Stieltjes transform.
2. We establish the infinite divisibility of the proposed compound distribution by constructing an auxiliary family of functions satisfying the defining properties of characteristic functions.
3. We further prove that the resulting characteristic function is uniformly continuous, positive definite, and has no real zeros.

The results extend previous characterizations of compound Poisson distributions with other infinitely divisible summands, such as geometric and Cauchy-variance distributions, to

the number-theoretic setting of the Riemann zeta distribution. In this way, the present study establishes a connection between the convolution structure of compound Poisson distributions and the arithmetic structure underlying the Riemann zeta distribution.

The remainder of this paper is organized as follows. Section 2 introduces the definitions and classical results used throughout the paper. Section 3 reviews the Riemann zeta distribution, derives its moments and characteristic function, and discusses its infinite divisibility through the von Mangoldt function. Section 4 presents the compound Poisson distribution and the infinite divisibility of the Poisson distribution. Section 5 contains the main theoretical results, including the characteristic function of the compound Poisson distribution with Riemann zeta summands and proofs of its continuity, positive definiteness, infinite divisibility, and non-vanishing properties. Section 6 provides numerical illustrations and comparisons with previous studies. Finally, Section 7 concludes the paper and discusses possible future research directions.

2. Preliminaries

In this section, we briefly recall several definitions and classical results that will be used throughout the paper.

Definition 2.1 (Characteristic function [6]). *The characteristic function of a random variable X is defined as*

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \int e^{itx} dF_X(x), \quad t \in \mathbb{R}. \quad (2)$$

Theorem 2.2 ([6]). *Every characteristic function φ_X satisfies the following properties: (i) $\varphi_X(0) = 1$; (ii) $|\varphi_X(t)| \leq 1$ for all $t \in \mathbb{R}$; (iii) $\varphi_X(-t) = \overline{\varphi_X(t)}$; and (iv) φ_X is uniformly continuous on $\mathbb{R}$.*

Definition 2.3 (Positive definiteness [6]). *A complex-valued function φ is positive definite if, for every $N \in \mathbb{N}$, $t_1, \dots, t_N \in \mathbb{R}$, and $c_1, \dots, c_N \in \mathbb{C}$,*

$$Q = \sum_{j=1}^N \sum_{l=1}^N c_j \overline{c_l} \varphi(t_j - t_l) \geq 0. \quad (3)$$

Theorem 2.4 (Bochner's criterion [6]). *A function $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ is a characteristic function if and only if it is continuous, positive definite, and satisfies $\varphi(0) = 1$.*

Definition 2.5 (Infinite divisibility [6]). *A distribution function F , or equivalently its characteristic function φ , is infinitely divisible if, for every positive integer m , there exists a characteristic function φ_m such that*

$$\varphi(t) = [\varphi_m(t)]^m. \quad (4)$$

Definition 2.6 (von Mangoldt function [4]). *The von Mangoldt function $\Lambda(n)$ is defined by*

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for a prime } p \text{ and integer } k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Definition 2.7 (Compound Poisson distribution [5, 16]). *Let $X_1, X_2, \dots$ be independent and identically distributed random variables, independent of a Poisson random variable N with mean λ . Then*

$$Z = X_1 + X_2 + \dots + X_N, \quad (6)$$

with the convention that $Z = 0$ when $N = 0$, is said to follow a compound Poisson distribution.

Theorem 2.8 ([5, 16]). If φ_X is the characteristic function of the summands X_i in Definition 2.7, then the characteristic function of Z is

$$\varphi_Z(t) = \exp\{\lambda [\varphi_X(t) - 1]\}, \quad (7)$$

which corresponds to an infinitely divisible distribution.

3. The Riemann Zeta Distribution and Its Infinite Divisibility

3.1. The Riemann zeta function and the Riemann zeta distribution

For a complex variable $s = \sigma + i\tau$ with $\sigma > 1$, the Riemann zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}, \quad \sigma > 1, \quad (8)$$

where the product is taken over all prime numbers p (the Euler product [17]). The series converges absolutely for $\sigma > 1$.

The Riemann zeta distribution is obtained from the zeta function through the transformation $X = \log n$ [4, 18]. Although the Riemann zeta function is defined for complex arguments, the probability mass function of the Riemann zeta distribution depends only on the real parameter σ .

Definition 3.1. A random variable X follows the Riemann zeta distribution with parameter $\sigma > 1$ if

$$P(X = \log n) = \frac{n^{-\sigma}}{\zeta(\sigma)}, \quad n = 1, 2, 3, \dots \quad (9)$$

The normalization condition follows directly from the definition of the Riemann zeta function, since

$$\sum_{n=1}^{\infty} \frac{n^{-\sigma}}{\zeta(\sigma)} = \frac{\zeta(\sigma)}{\zeta(\sigma)} = 1. \quad (10)$$

Figure 1 illustrates the probability mass function (9) for several values of σ . As σ increases, the distribution concentrates increasingly on $n = 1$, whereas for σ close to the boundary $\sigma > 1$ the mass spreads out over a much heavier tail, reflecting the slower decay of $n^{-\sigma}$ and the divergence of $\zeta(\sigma)$ as $\sigma \rightarrow 1^+$.

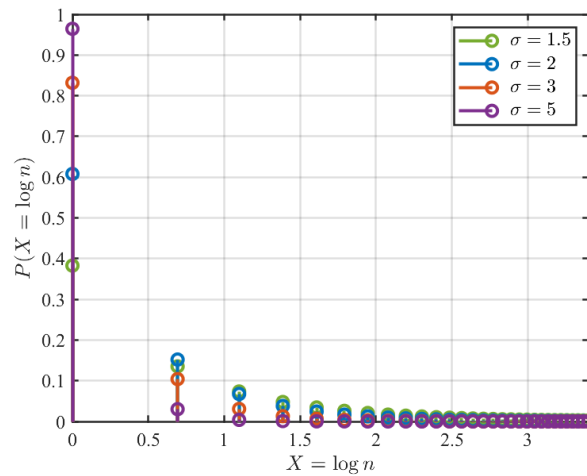


Figure 1. Probability mass function of the Riemann zeta distribution, $P(X = \log n) = n^{-\sigma}/\zeta(\sigma)$, for $\sigma = 1.5, 2, 3, 5$ and $n = 1, \dots, 30$.

3.2. Moment generating function, characteristic function, mean and variance

Theorem 3.2. *Let $X = \log n$ follow the Riemann zeta distribution in Definition 3.1. Then, for $u < \sigma - 1$, the moment generating function is*

$$M_X(u) = \mathbb{E} [e^{uX}] = \frac{\zeta(\sigma - u)}{\zeta(\sigma)}, \quad (11)$$

and the characteristic function is

$$\varphi_X(u) = \mathbb{E} [e^{iuX}] = \frac{\zeta(\sigma - iu)}{\zeta(\sigma)}. \quad (12)$$

Moreover,

$$\mathbb{E}[X] = -\frac{\zeta'(\sigma)}{\zeta(\sigma)}, \quad \text{Var}(X) = \frac{\zeta''(\sigma)}{\zeta(\sigma)} - \left(\frac{\zeta'(\sigma)}{\zeta(\sigma)} \right)^2. \quad (13)$$

Proof. Since $X = \log n$, we have $e^{uX} = n^u$. Therefore,

$$M_X(u) = \sum_{n=1}^{\infty} n^u \frac{n^{-\sigma}}{\zeta(\sigma)} = \frac{1}{\zeta(\sigma)} \sum_{n=1}^{\infty} n^{-(\sigma-u)} = \frac{\zeta(\sigma - u)}{\zeta(\sigma)}. \quad (14)$$

Replacing u by iu gives the characteristic function. Differentiating the moment generating function yields

$$M'_X(u) = -\frac{\zeta'(\sigma - u)}{\zeta(\sigma)}, \quad (15)$$

and

$$M''_X(u) = \frac{\zeta''(\sigma - u)}{\zeta(\sigma)}. \quad (16)$$

Evaluating these expressions at $u = 0$ gives the mean and variance formulas. ■

3.3. Positive definiteness and uniform continuity

Proposition 3.3. *The characteristic function φ_X in (12) is positive definite and uniformly continuous.*

Proof. For $u_1, u_2 \in \mathbb{R}$, we have

$$|\varphi_X(u_1) - \varphi_X(u_2)| = \frac{1}{\zeta(\sigma)} \left| \sum_{n=1}^{\infty} n^{-\sigma} (n^{iu_1} - n^{iu_2}) \right|. \quad (17)$$

Since $\sum_{n=1}^{\infty} n^{-\sigma}$ converges for $\sigma > 1$ and each term is bounded, the dominated convergence theorem implies that the right-hand side approaches zero as $u_1 \rightarrow u_2$. Hence, φ_X is uniformly continuous.

For positive definiteness, let $N \in \mathbb{N}$, $u_1, \dots, u_N \in \mathbb{R}$, and $c_1, \dots, c_N \in \mathbb{C}$. Then,

$$\sum_{j,l=1}^N c_j \bar{c}_l \varphi_X(u_j - u_l) = \frac{1}{\zeta(\sigma)} \sum_{n=1}^{\infty} n^{-\sigma} \left| \sum_{j=1}^N c_j n^{iu_j} \right|^2 \geq 0. \quad (18)$$

Therefore, φ_X is positive definite. ■

3.4. The von Mangoldt transformation and infinite divisibility

The infinite divisibility of the Riemann zeta distribution follows from the representation of $\log \zeta(\sigma)$ through the von Mangoldt function. From the Euler product,

$$\log \zeta(\sigma) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma}, \quad \sigma > 1. \quad (19)$$

Consequently, using (12),

$$\varphi_X(u) = \exp \left\{ \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma} \log n} (n^{iu} - 1) \right\}. \quad (20)$$

The coefficients

$$\frac{\Lambda(n)}{n^{\sigma} \log n} \quad (21)$$

are non-negative and summable for $\sigma > 1$. Hence, (20) has the Lévy–Khintchine representation of an infinitely divisible characteristic function.

Theorem 3.4 ([2, 3]). *The characteristic function of the Riemann zeta distribution is infinitely divisible.*

Proof sketch. For every positive integer m , define

$$\varphi_m(u) = \exp \left\{ \frac{1}{m} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma} \log n} (n^{iu} - 1) \right\}. \quad (22)$$

Each φ_m corresponds to a Lévy–Khintchine representation with a non-negative measure. Therefore, it is a characteristic function. Furthermore,

$$[\varphi_m(u)]^m = \varphi_X(u), \quad (23)$$

which proves the infinite divisibility of φ_X . ■

4. The Compound Poisson Distribution

4.1. Basic properties

Let N be a Poisson random variable with parameter $\lambda > 0$, and let $X_1, X_2, \dots$ be independent and identically distributed random variables independent of N . Define the compound Poisson random variable

$$S = X_1 + \dots + X_N, \quad (24)$$

with the convention that $S = 0$ when $N = 0$. Using conditioning on N , the first two moments of S are given by [16]

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X], \quad (25)$$

$$\text{Var}(S) = \mathbb{E}[N]\text{Var}(X) + \text{Var}(N)(\mathbb{E}[X])^2. \quad (26)$$

Moreover, the characteristic function of S satisfies

$$\varphi_S(t) = \exp \{ \lambda(\varphi_X(t) - 1) \}. \quad (27)$$

4.2. The Poisson distribution and its infinite divisibility

Definition 4.1 ([21]). A random variable N follows the Poisson distribution with parameter $\lambda > 0$ if

$$P(N = n) = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0, 1, 2, \dots \quad (28)$$

The mean and variance of a Poisson random variable are both equal to λ . Its moment generating function and characteristic function are

$$M_N(t) = \exp\{\lambda(e^t - 1)\}, \quad (29)$$

and

$$\varphi_N(t) = \exp\{\lambda(e^{it} - 1)\}, \quad (30)$$

respectively.

Proposition 4.2. *The Poisson distribution is infinitely divisible.*

Proof. For any positive integer m , let N_m be a Poisson random variable with parameter λ/m . Its characteristic function is

$$\varphi_{N_m}(t) = \exp\left\{\frac{\lambda}{m}(e^{it} - 1)\right\}. \quad (31)$$

Therefore,

$$[\varphi_{N_m}(t)]^m = \exp\{\lambda(e^{it} - 1)\} = \varphi_N(t), \quad (32)$$

which proves the infinite divisibility of the Poisson distribution. $\blacksquare$

Combining Theorem 2.8 with Proposition 4.2, every compound Poisson distribution is infinitely divisible. In the following section, we specialize this result to the case where the summands follow the Riemann zeta distribution introduced in Section 3 and investigate the resulting characteristic function.

5. The Compound Poisson Distribution as the Sum of Riemann Zeta Distributions

Let X follow the Riemann zeta distribution with parameter $\sigma > 1$, and let $X_1, X_2, \dots$ be independent copies of X . Let N be a Poisson random variable with parameter $\lambda > 0$, independent of the X_i . Define the compound Poisson random variable

$$S = X_1 + X_2 + \dots + X_N. \quad (33)$$

5.1. Moments and transforms

Proposition 5.1. *The mean and variance of S in (33) are*

$$\mathbb{E}[S] = -\lambda \frac{\zeta'(\sigma)}{\zeta(\sigma)}, \quad (34)$$

$$\text{Var}(S) = \lambda \frac{\zeta''(\sigma)}{\zeta(\sigma)}. \quad (35)$$

Proof. Since N follows a Poisson distribution,

$$\mathbb{E}[N] = \text{Var}(N) = \lambda.$$

Using the compound Poisson moment formulas in Section 4 and the moments of the Riemann zeta distribution in (13), we obtain

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = -\lambda \frac{\zeta'(\sigma)}{\zeta(\sigma)}.$$

Moreover,

$$\text{Var}(S) = \mathbb{E}[N]\text{Var}(X) + \text{Var}(N)(\mathbb{E}[X])^2.$$

Since

$$\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2 = \frac{\zeta''(\sigma)}{\zeta(\sigma)},$$

the variance reduces to

$$\text{Var}(S) = \lambda \frac{\zeta''(\sigma)}{\zeta(\sigma)}.$$

■

Proposition 5.2. *The moment generating function and characteristic function of S are given by*

$$M_S(u) = \exp \left\{ \lambda \left[\frac{\zeta(\sigma - u)}{\zeta(\sigma)} - 1 \right] \right\}, \quad (36)$$

$$\varphi_S(u) = \exp \left\{ \lambda \left[\frac{\zeta(\sigma - iu)}{\zeta(\sigma)} - 1 \right] \right\}. \quad (37)$$

Proof. The results follow directly by substituting the moment generating function and the characteristic function of the Riemann zeta distribution, (11) and (12), into the compound Poisson formula. ■

5.2. Basic properties of the characteristic function

Proposition 5.3. *The characteristic function φ_S in (37) satisfies*

1. $\varphi_S(0) = 1$;
2. $|\varphi_S(u)| \leq 1$ for all $u \in \mathbb{R}$;
3. $\varphi_S(-u) = \overline{\varphi_S(u)}$.

Proof. Setting $u = 0$ in (37) gives

$$\varphi_S(0) = \exp\{\lambda(1 - 1)\} = 1.$$

Since

$$\varphi_S(u) = \exp\{\lambda(\varphi_X(u) - 1)\},$$

and φ_X is a characteristic function, we have

$$|\varphi_X(u)| \leq 1.$$

Therefore,

$$\operatorname{Re}(\varphi_X(u) - 1) \leq 0,$$

which implies

$$|\varphi_S(u)| \leq 1.$$

Finally,

$$\varphi_X(-u) = \overline{\varphi_X(u)}$$

implies

$$\varphi_S(-u) = \overline{\varphi_S(u)}.$$

■

5.3. Infinite divisibility

For every positive integer m , define

$$\varphi_{S_m}(u) = \exp \left\{ \frac{\lambda}{m} \left[\frac{\zeta(\sigma - iu)}{\zeta(\sigma)} - 1 \right] \right\}. \quad (38)$$

Proposition 5.4. *For every positive integer m , the function φ_{S_m} in (38) is a characteristic function.*

Proof. The function φ_{S_m} is the characteristic function of a compound Poisson distribution with intensity parameter λ/m and summand characteristic function φ_X . Hence, by the compound Poisson characterization in Section 4, φ_{S_m} is a valid characteristic function. ■

Theorem 5.5. *The compound Poisson distribution with characteristic function φ_S given by (37) is infinitely divisible.*

Proof. For every positive integer m ,

$$[\varphi_{S_m}(u)]^m = \exp \left\{ \lambda \left[\frac{\zeta(\sigma - iu)}{\zeta(\sigma)} - 1 \right] \right\} = \varphi_S(u). \quad (39)$$

Since φ_{S_m} is a characteristic function by Proposition 5.4, Definition 2.5 is satisfied. Therefore, φ_S is infinitely divisible. ■

5.4. Absence of real zeros

Theorem 5.6. *The characteristic function φ_S has no real zeros.*

Proof. From (37),

$$\varphi_S(u) = \exp \{ \lambda(\varphi_X(u) - 1) \}.$$

The quantity $\lambda(\varphi_X(u) - 1)$ is finite for every $u \in \mathbb{R}$ because $|\varphi_X(u)| \leq 1$. Since the exponential function never vanishes on the complex plane,

$$\varphi_S(u) \neq 0, \quad u \in \mathbb{R}.$$

Therefore, the characteristic function has no real zeros. ■

Remark 5.7. *The absence of real zeros follows directly from the exponential representation of the compound Poisson characteristic function. This property is a consequence of the infinitely divisible structure of compound Poisson distributions and holds for the Riemann zeta summand considered in this work.*

The results above provide a complete characterization of the compound Poisson distribution generated by Riemann zeta random variables, including its moments, characteristic function, infinite divisibility, and the non-vanishing property of its characteristic function.

6. Numerical Illustration and Discussion

6.1. Numerical illustration

Because the mean and variance of X in (13) depend on σ through $\zeta(\sigma)$, $\zeta'(\sigma)$, and $\zeta''(\sigma)$, Table 1 reports numerical values of $\zeta(\sigma)$, $\mathbb{E}[X]$, and $\text{Var}[X]$ for several values of σ . The corresponding compound Poisson moments $\mathbb{E}[S]$ and $\text{Var}[S]$ are also reported for $\lambda = 5$, obtained from Proposition 5.1.

Table 1. Numerical values of $\zeta(\sigma)$, $\mathbb{E}[X]$, $\text{Var}[X]$, and the corresponding compound Poisson moments $\mathbb{E}[S]$ and $\text{Var}[S]$ for $\lambda = 5$.

σ	$\zeta(\sigma)$	$\mathbb{E}[X]$	$\text{Var}[X]$	$\mathbb{E}[S]$	$\text{Var}[S]$
2	1.644934	0.569961	0.884482	2.849805	4.422410
3	1.202057	0.164823	0.172281	0.824115	0.861405
4	1.082323	0.063670	0.056056	0.318350	0.280280
5	1.036928	0.027556	0.022127	0.137780	0.110635

Two features of Table 1 are worth noting. First, $\mathbb{E}[X]$ and $\text{Var}[X]$ decrease monotonically as σ increases. This behavior reflects the concentration of the probability mass function

$$P(X = \log n) = \frac{n^{-\sigma}}{\zeta(\sigma)}$$

around $n = 1$, which corresponds to $X = 0$, when σ becomes large. Indeed, as $\sigma \rightarrow \infty$, the Riemann zeta function satisfies $\zeta(\sigma) \rightarrow 1$, and the distribution degenerates at zero. Consequently, the compound Poisson random variable S also converges to zero.

Second, for a fixed value of σ , both $\mathbb{E}[S]$ and $\text{Var}[S]$ vary linearly with the Poisson parameter λ , as shown in (34)–(35). This linear dependence follows from the compound Poisson structure and the fact that the moments of the summand distribution do not depend on λ .

Figure 2 plots the modulus $|\varphi_S(u)|$ of the characteristic function in (37) for $\lambda = 5$ and $\sigma = 2, 3, 5$, over $u \in [-20, 20]$. The plot provides a direct numerical illustration of Theorem 5.6, for every σ shown, $|\varphi_S(u)|$ remains strictly positive throughout the plotted range, approaching but never reaching zero. This behavior is most visible for $\sigma = 2$, where the summand characteristic function φ_X exhibits the largest variation in magnitude.

The curve is symmetric about $u = 0$, which is consistent with Proposition 5.3(iii), and satisfies $|\varphi_S(0)| = 1$ for every σ , in agreement with Proposition 5.3(i). As σ increases, $|\varphi_S(u)|$ remains closer to one over the entire range of u , reflecting the decrease in $\text{Var}[X]$ reported in Table 1.

6.2. Discussion: comparison with related compound Poisson characterizations

The characterization developed in this work follows the general approach used in previous studies of compound Poisson distributions with specific summand distributions. In particular,

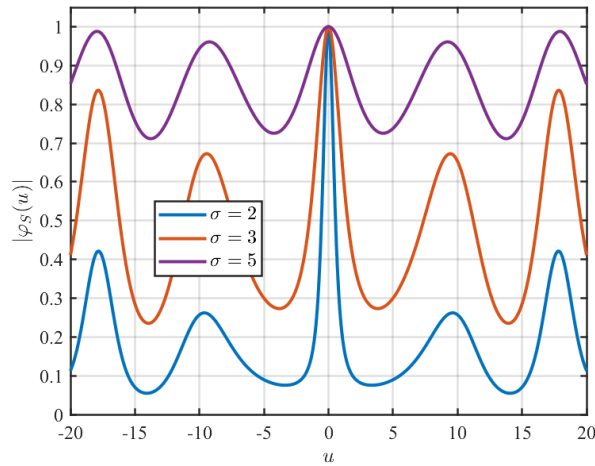


Figure 2. Modulus of the characteristic function $\varphi_S(u) = \exp\{\lambda[\zeta(\sigma - iu)/\zeta(\sigma) - 1]\}$ of the compound Poisson distribution with Riemann zeta summands for $\sigma = 2, 3, 5$ and $\lambda = 5$. The curve remains strictly positive over the plotted range, illustrating the non-vanishing property established in Theorem 5.6

the characteristic function of the summand is first derived, followed by the construction of the compound Poisson characteristic function and the investigation of its fundamental properties, including continuity, positive definiteness, and infinite divisibility.

The present setting differs from classical examples through the special role of the Riemann zeta function. The characteristic function of the Riemann zeta summand is expressed in the closed form

$$\varphi_X(u) = \frac{\zeta(\sigma - iu)}{\zeta(\sigma)}, \quad (40)$$

which establishes a direct connection between the compound Poisson distribution and quantities from analytic number theory. In particular, the moments of the resulting distribution can be represented through the logarithmic derivatives of the Riemann zeta function,

$$\frac{\zeta'(\sigma)}{\zeta(\sigma)} \quad \text{and} \quad \frac{\zeta''(\sigma)}{\zeta(\sigma)}.$$

Such a representation provides a number-theoretic interpretation of the compound Poisson model considered in this paper.

Furthermore, the characteristic function

$$\varphi_S(u) = \exp\{\lambda(\varphi_X(u) - 1)\} \quad (41)$$

provides an immediate explanation for the absence of real zeros established in Theorem 5.6. Since the exponential function is non-zero everywhere on the complex plane, the characteristic function of this compound Poisson distribution cannot vanish.

This observation is not specific to the Riemann zeta summand but follows from the general exponential representation of compound Poisson characteristic functions. The Riemann zeta distribution therefore provides a particular number-theoretic example within the broader class of infinitely divisible compound Poisson distributions.

7. Conclusion

This paper investigated the compound Poisson distribution $S = X_1 + \cdots + X_N$, where N is a Poisson random variable and the summands X_i are independent and identically distributed Riemann zeta random variables with parameter $\sigma > 1$. By combining the characteristic function of the Riemann zeta distribution with the compound Poisson construction, we obtained the characteristic function

$$\varphi_S(u) = \exp \left\{ \lambda \left[\frac{\zeta(\sigma - iu)}{\zeta(\sigma)} - 1 \right] \right\}. \quad (42)$$

We showed that this characteristic function satisfies the fundamental properties of a characteristic function, including normalization, boundedness, and conjugate symmetry.

Using the infinite divisibility representation of the Riemann zeta distribution through the von Mangoldt function, together with the compound Poisson structure, we constructed an appropriate family of characteristic functions whose powers reproduce φ_S . This establishes the infinite divisibility of the compound Poisson distribution generated by Riemann zeta summands. Furthermore, we proved that its characteristic function has no real zeros, which follows directly from its exponential representation.

The present work focuses on the theoretical characterization of this distribution and does not investigate potential applications. Several directions remain open for future study. From the theoretical perspective, it would be of interest to derive a more explicit Lévy–Khinchine representation for the resulting infinitely divisible distribution and to investigate the corresponding Lévy process. From the applied perspective, since compound Poisson models play an important role in areas such as aggregate claim modeling and risk theory [5, 6], further research may explore whether the arithmetic structure inherited from the Riemann zeta distribution can provide useful extensions or alternative formulations in stochastic modeling and related applications.

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