



ON THE DIRECT SUM OF GENERALIZED MODULES AND ITS PROPERTIES

MUHAMMAD ALI RIZQAN*

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Lambung Mangkurat University, Banjarbaru, Kalimantan Selatan, Indonesia

*Corresponding Author: alirizqanmuhammad@gmail.com

ABSTRACT

The development of generalized algebraic structure theory currently focuses on extending the concepts of classical module theory over rings with unity. However, studies regarding the construction and elementary properties of the external direct sum of generalized modules remain underexplored. This research aims to examine the structure of the external direct sum of generalized modules along with the existence of its homomorphic mappings. The method employed is the axiomatic deductive method by analyzing the governing axioms in generalized module and group theories. The main contribution of this work lies in establishing the first formal framework for the external direct sum structure under strict generalized module axioms, alongside proving its novel morphic properties. The results demonstrate that the external direct sum of generalized modules inherently satisfies the axioms of generalized modules and allows for the construction of corresponding submodule structures. Furthermore, this study successfully constructs new morphic mappings via canonical projections and induced mappings from component homomorphisms, which are proven to be generalized module epimorphic. These findings strengthen the structural foundation of generalized module theory and offer potential avenues for further development in generalized factor module constructions.

Keywords: Epimorphism, External Direct Sum, Generalized Module, Homomorphism, Submodule.

1. Introduction

The development of non-classical algebraic structure theory has advanced significantly since the introduction of the generalized group concept by Adéníran et al. [1]. This theory subsequently extended to other algebraic systems, as seen in the generalized semigroup concept by Fatehi and Molaei [2]. The axiomatization of these generalized structures was then extended to multiple-operation systems when Fatehi and Molaei [3] formulated generalized rings, which was simultaneously accompanied by the development of generalized actions by Maleki and Molaei [4]. Building upon these ring developments, Romeo and Sneha [5] constructed the theory of generalized modules, encompassing the definitions of submodules up to generalized module homomorphisms.

Research regarding the abstract properties of these structures has grown rapidly in recent years. Aldwoah [6] examined more deeply the elementary properties and subgroup structures of generalized groups. Furthermore, the topological and semi-topological aspects of these generalized algebraic systems were explored by Jehangir and Khan [7], who investigated generalized s-topological groups. By the year 2024, the dynamics of generalized algebraic research experienced a major expansion across various branches, including the investigation of

finite generalized group characteristics by Shajareh and Poursalavati [8], the analysis of elementary properties of generalized rings by Rosyadi [9], and the development of generalized crossed modules over generalized groups-groupoids by Akiz [10].

Alongside theoretical advancements, practical implementation was also achieved by Jaiyeola and Smarandache [11], who successfully applied generalized group theory to cryptography. This module framework was most recently expanded by Rizqan [12], who extended generalized modules into fully generalized modules, while order-theoretic and topological compatibility in generalized group structures was advanced by Gracia Pacheco et al. [13]. Although the foundational theory of generalized modules has been firmly established, the scope of this framework remains highly open for further extension, particularly focusing on the framework introduced by Romeo and Sneha [5]. In classical module theory, the concept of the external direct sum plays a crucial role as a method for constructing new modules from existing ones. However, to the best of our knowledge, a formalization of how the external direct sum structure behaves under the strict axioms of generalized modules has not yet been formulated.

Therefore, this paper aims to establish the formal structure of the external direct sum of generalized modules and rigorously investigate its elementary properties. This study focuses on analyzing operational closure, the existence of corresponding submodules, and the behavior of canonical homomorphic and epimorphic mappings. Through this deductive axiomatic approach, the obtained results are expected to strengthen the structural foundation of generalized module theory, while simultaneously paving the way for future developments in generalized category theory.

2. Preliminaries

In this section, we present several definitions and fundamental concepts regarding generalized group, generalized modules and generalized module homomorphisms. These structured definitions will serve as the axiomatic foundation throughout the subsequent discussions in this paper.

Definition 2.1 A generalized group is a non-empty set admitting a binary operation called multiplication subject to the set of rules given below.

1. $(xy)z = x(yz)$ for all $x, y, z \in G$.
2. For each $x \in G$, there exists a unique $e(x) \in G$ such that $xe(x) = x = e(x)x$
3. For each $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = e(x) = x^{-1}x$

Definition 2.2 Let R be a ring with unity 1_R . A generalized group M is said to be a (left) generalized R -module if there exists an action $R \times M \rightarrow M$ denoted by $(r, \alpha) \mapsto r\alpha$ such that for all $r, s \in R$ and $\alpha, \beta \in M$, the following conditions hold:

1. $(r + s)\alpha = r\alpha + s\alpha$
2. $r(\alpha + \beta) = r\alpha + r\beta$
3. $r(s\alpha) = (rs)\alpha$
4. $re(\alpha) = e(\alpha)$ where $e(\alpha) \in M$ is the identity element of α
5. $1_R\alpha = \alpha$

Based on the definition of a generalized module, the structural characteristics of its sub-objects can be specified. We define a generalized submodule as follows.

Definition 2.3 Let M be a generalized R -module and $N \subset M$ is said to be generalized submodule of M if N is a generalized subgroup of G and for each $r \in R$ dan $\alpha \in N$, $r\alpha \in N$.

The following definition provides the rigorous formulation of a generalized module homomorphism.

Definition 2.4 Let M and N be two generalized R -modules. A function $f : M \rightarrow N$ is called generalized module homomorphism if

$$\begin{aligned} f(\alpha + \beta) &= f(\alpha) + f(\beta), & \text{for all } \alpha, \beta \in M \\ f(r\alpha) &= rf(\alpha), & \text{for all } r \in R, \alpha \in M \end{aligned}$$

Next, we state the definitions of generalized module epimorphisms.

Definition 2.5 Let M and N be two generalized R -modules and let $f : M \rightarrow N$ be a generalized module homomorphism. The mapping f is called a generalized module epimorphism if f is surjective, that is, for every $\alpha, \beta \in N$, there exists an element $\alpha \in M$ such that $f(\alpha) = \beta$.

3. Results and Discussion

In this section, we extend the structural framework of generalized modules by defining their external direct sum and establishing its elementary algebraic properties.

3.1. Direct Sum of Generalized Modules

Two generalized R -modules M_1 dan M_2 can be constructed into a new generalized R -module denoted by $M_1 \oplus M_2$. This new structure preserves the algebraic characteristics of M_1 and M_2 .

Definition 3.1 Let R be a ring with an identity element 1_R , and let M_1 and M_2 be two generalized R -modules. The external direct sum of M_1 and M_2 , denoted by $M_1 \oplus M_2$, is the set of all ordered pairs:

$$M_1 \oplus M_2 = \{(\alpha, \beta) \mid \alpha \in M_1, \beta \in M_2\}$$

This set forms a (left) generalized R -module via the addition and scalar multiplication operations defined component-wise as follows:

$$\begin{aligned} (\alpha, \beta) + (\theta, \gamma) &= (\alpha +_{M_1} \theta, \beta +_{M_2} \gamma) \\ r(\alpha, \beta) &= (r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) \end{aligned}$$

for every $(\alpha, \beta), (\theta, \gamma) \in M_1 \oplus M_2$ and $r \in R$.

The following is an example of the generalized R -module $M_1 \oplus M_2$.

Example 3.2 Let $\mathbb{Z}_{2n}$ and $\mathbb{Z}_{2n-1}$ be two generalized modules over the ring $\mathbb{Z}$ for some $n \in \mathbb{N}$. The external direct sum $\mathbb{Z}_{2n} \oplus \mathbb{Z}_{2n-1}$ with component-wise operations:

$$\begin{aligned} (\alpha, \beta) + (\theta, \gamma) &= (\alpha +_{M_1} \theta, \beta +_{M_2} \gamma) = (\alpha, \gamma) \\ r(\alpha, \beta) &= (r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) = (\alpha, \beta) \end{aligned}$$

is a generalized module over the ring $\mathbb{Z}$.

In the generalized R -module $M_1 \oplus M_2$, the elementary properties related to the identity element, inverse element, and scalar multiplication are preserved.

Theorem 3.3 If $M_1 \oplus M_2$ is an external sum of generalized R -modules, then for every $(\alpha, \beta) \in M_1 \oplus M_2$ and $r \in R$, the following elementary properties hold:

$$1. \quad e_{M_1 \oplus M_2}(\alpha, \beta) = (e_{M_1}(\alpha), e_{M_2}(\beta))$$

2. $-(\alpha, \beta) = (-\alpha, -\beta)$
3. $re_{M_1 \oplus M_2}(\alpha, \beta) = e_{M_1 \oplus M_2}(\alpha, \beta)$
4. $re_{M_1 \oplus M_2}(\alpha, \beta) = e_{M_1 \oplus M_2}(r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta)$

PROOF.

(1) By Definition 3.1 $M_1 \oplus M_2$ is a generalized R -module. Based on Definition 2.1, we obtain

$$(\alpha, \beta) + e_{M_1 \oplus M_2}(\alpha, \beta) = (\alpha, \beta) = e_{M_1 \oplus M_2}(\alpha, \beta) + (\alpha, \beta).$$

Let $e_{M_1 \oplus M_2}(\alpha, \beta) = (\xi, \mu)$. Then the equation can be written as $(\alpha +_{M_1} \xi, \beta +_{M_2} \mu) = (\alpha, \beta) = (\xi +_{M_1} \alpha, \mu +_{M_2} \beta)$. By Definition 2.1, it yields $\xi = e_{M_1}(\alpha)$ and $\mu = e_{M_2}(\beta)$, which implies $e_{M_1 \oplus M_2}(\alpha, \beta) = (e_{M_1}(\alpha), e_{M_2}(\beta))$.

(2) Based on Definition 3.1, Definition 2.1, and Theorem 3.3 (1), it holds that

$$(\alpha, \beta) - (\alpha, \beta) = (e_{M_1}(\alpha), e_{M_2}(\beta)) = -(\alpha, \beta) + (\alpha, \beta).$$

Let $-(\alpha, \beta) = (\xi, \mu)$. Which yields

$$(\alpha +_{M_1} \xi, \beta +_{M_2} \mu) = (e_{M_1}(\alpha), e_{M_2}(\beta)) = (\xi +_{M_1} \alpha, \mu +_{M_2} \mu).$$

By Definition 2.2, we obtain $\xi = -\alpha$ and $\mu = -\beta$, so that $-(\alpha, \beta) = (-\alpha, -\beta)$.

(3) Based on Theorem 3.3 (1), Definition 3.1 and Definition 2.1, for any scalar $r \in R$ and local identity element $e_{M_1 \oplus M_2}(\alpha, \beta) \in M_1 \oplus M_2$, we obtain

$$re_{M_1 \oplus M_2}(\alpha, \beta) = r(e_{M_1}(\alpha), e_{M_2}(\beta)) = (r \cdot_{M_1} e_{M_1}(\alpha), r \cdot_{M_2} e_{M_2}(\beta)) = (e_{M_1}(\alpha), e_{M_2}(\beta))$$

(4) By Theorem 3.3 (1), Definition 3.1, and Theorem 3 we obtain

$$\begin{aligned} re_{M_1 \oplus M_2}(\alpha, \beta) &= r(e_{M_1}(\alpha), e_{M_2}(\beta)) \\ &= (r \cdot_{M_1} e_{M_1}(\alpha), r \cdot_{M_2} e_{M_2}(\beta)) \\ &= (e_{M_1}(r \cdot_{M_1} \alpha), e_{M_2}(r \cdot_{M_2} \beta)) \\ &= e_{M_1 \oplus M_2}(r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) \end{aligned} \quad \blacksquare$$

To facilitate our discussion, we first introduce the definition of a cyclic generalized module as follows.

Definition 3.4 Let M be a generalized R -module. M is called a cyclic generalized module if there exists an element $\xi \in M$, called a generator of M , such that $M = R\xi = \{r\xi \mid r \in R\}$.

If the generalized module is cyclic, then the external direct sum of generalized modules reduces to an ordinary module.

Theorem 3.5 Let $M_1 \oplus M_2$ be an external direct sum of generalized R -modules. If $M_1 \oplus M_2$ is a cyclic module, that is, there exists a generator $(\xi, \mu) \in M_1 \oplus M_2$ such that $M_1 \oplus M_2 = R(\xi, \mu)$, then $M_1 \oplus M_2$ is a R -module.

PROOF. It is given that $M_1 \oplus M_2 = R(\xi, \mu)$ for some $(\xi, \mu) \in M_1 \oplus M_2$. Based on the properties of canonical projection mapping onto the component coordinates of the Cartesian product and Definition 3.4, this condition implies that its constituent components are also cyclic, namely $M_1 = R\xi$ and $M_2 = R\mu$ respectively for $\xi \in M_1$ and $\mu \in M_2$.

According to Theorem 5 by G.Romeo & Sneha [5] and Definition 3.4, $M_1 = R\xi$ and $M_2 = R\mu$ are R -modules. Consequently, since $M_1 \oplus M_2 = R\xi \oplus R\mu$ is constructed from two modules, by Definition 3.1 it follows that $M_1 \oplus M_2$ is a R -module. ■

An external direct sum of generalized R -modules is a normal generalized group.

Theorem 3.6 If $M_1 \oplus M_2$ is an external direct sum of generalized R -modules, then $M_1 \oplus M_2$ is a normal generalized group.

PROOF. Based on Definition 3.1, it is clear that $M_1 \oplus M_2$ constitutes a generalized R -module, and by Definition 2.2, $M_1 \oplus M_2$ is a generalized group. Since R is a ring with unity, there exists an element $0_R \in R$ such that $0_R +_R \alpha = \alpha = \alpha +_R 0_R$ for every $\alpha \in R$. According to Definition 3.1, Definition 2.2, and Theorem 3.3 we obtain:

$$\begin{aligned} 0_R(\alpha, \beta) &= (r - r)(\alpha, \beta) \\ &= r(\alpha, \beta) - r(\alpha, \beta) \\ &= (r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) - (r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) \\ &= e_{M_1 \oplus M_2}(r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) \\ &= e_{M_1 \oplus M_2}(\alpha, \beta) \end{aligned}$$

Consequently, for $e_{M_1 \oplus M_2}((\alpha, \beta) + (\theta, \gamma)) \in M_1 \oplus M_2$ we have:

$$\begin{aligned} e_{M_1 \oplus M_2}((\alpha, \beta) + (\theta, \gamma)) &= 0_R((\alpha, \beta) + (\theta, \gamma)) \\ &= 0_R(\alpha, \beta) + 0_R(\theta, \gamma) \\ &= e_{M_1 \oplus M_2}(\alpha, \beta) + e_{M_1 \oplus M_2}(\theta, \gamma) \end{aligned}$$

Thus, the map of local identities distributes over the addition, completing the proof that $M_1 \oplus M_2$ is a normal generalized group. ■

The next concept to be introduced in this paper is the full external direct sum of generalized modules along with its elementary properties.

3.2. Generalized Submodules of External Direct Sums Of Generalized Modules

The internal sub-structural properties of the constituent modules are directly inherited by their external direct sum representation, as stated in the following theorem:

Theorem 3.7 Let M_1 and M_2 be generalized R -modules. If N_1 is a generalized submodule of M_1 and N_2 is a generalized submodule of M_2 , then the external direct sum $N_1 \oplus N_2$ is a generalized submodule of $M_1 \oplus M_2$.

PROOF. Since N_1 and N_2 are generalized submodules of M_1 and M_2 respectively, it is clear that $N_1 \oplus N_2 \subseteq M_1 \oplus M_2$ and $N_1 \oplus N_2 \neq \emptyset$. Based on Definition 3.1 and Theorem 3.3 (2), for any $(a, b) \in N_1 \oplus N_2$, we obtain:

$$(\alpha, \beta) - (\theta, \gamma) = (\alpha, \beta) + (-\theta, -\gamma) = (\alpha +_{M_1} (-\theta), \beta +_{M_2} (-\gamma)) \in N_1 \oplus N_2.$$

Furthermore, for any $r \in R$, we obtain

$$r(\alpha, \beta) = (r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) \in N_1 \oplus N_2$$

Hence, $N_1 \oplus N_2$ is a generalized submodule of $M_1 \oplus M_2$. ■

The following is an example of an external direct sum of generalized submodules.

Example 3.8 The external direct sum of the generalized submodules $\mathbb{Z}_{4n}$ and $\mathbb{Z}_{5n-1}$ of the generalized module in Example 3.2 is a generalized submodule.

The following theorem explains that it is always possible to construct a generalized submodule which is the external direct sum of two trivial submodules.

Theorem 3.9 If $M_1 \oplus M_2$ is an external direct sum of generalized R -modules, then there exist a trivial submodule $E_1 \oplus E_2$ with $E_1 = \{e_{M_1}(a) | a \in M_1\}$ and $E_2 = \{e_{M_2}(a) | a \in M_2\}$, which are the sets of identity elements of M_1 and M_2 , respectively.

PROOF. Based on Proposition 2 by G. Romeo & Sneha [5], there must exist trivial submodules $E_1 = \{e_{M_1}(a) | a \in M_1\}$ of M_1 and $E_2 = \{e_{M_2}(a) | a \in M_2\}$ of M_2 . By Theorem 3.5, it is clear that $E_1 \oplus E_2$ is a generalized submodule of $M_1 \oplus M_2$ ■

The next concept to be introduced in this section is the framework of generalized module homomorphisms, along with their elementary structural properties.

3.3. Generalized Modules Homomorphism of External Direct Sums of Generalized Modules

In order to analyze how the structural properties of the external direct sum relate back to its constituent components, we define the canonical projection mappings and establish their epimorphic properties in the following definition and theorem.

Theorem 3.10 Let $M_1 \oplus M_2$ be an external direct sum of generalized R -modules. The canonical projection mappings $\pi_1 : M_1 \oplus M_2 \rightarrow M_1$ defined by $\pi_1(\alpha, \beta) = \alpha$ and $\pi_2 : M_1 \oplus M_2 \rightarrow M_2$ defined by $\pi_2(\alpha, \beta) = \beta$ are surjective generalized module homomorphisms (epimorphisms).

Proof. First, we show that π_1 is well defined. Let $(\alpha, \beta), (\theta, \gamma) \in M_1 \oplus M_2$ such that $(\alpha, \beta) = (\theta, \gamma)$. By the definition of equality in a cartesian product, it follows that $\alpha = \theta$ and $\beta = \gamma$. Thus, we obtain:

$$\pi_1(\alpha, \beta) = \alpha = \theta = \pi_1(\theta, \gamma)$$

Hence, π_1 is well-defined. Next, we show that π_1 preserves algebraic operations. Based on Definition 3.1 for any $(\alpha, \beta), (\theta, \gamma) \in M_1 \oplus M_2$, we have:

$$\pi_1((\alpha, \beta) + (\theta, \gamma)) = \pi_1(\alpha +_{M_1} \theta, \beta +_{M_2} \gamma) = \alpha +_{M_1} \theta = \pi_1(\alpha, \beta) +_{M_1} \pi_1(\theta, \gamma)$$

Furthermore, for any $r \in R$ and $(\alpha, \beta) \in M_1 \oplus M_2$, we obtain:

$$r\pi_1(\alpha, \beta) = r \cdot_{M_1} \alpha = \pi_1(r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) = \pi_1(r(\alpha, \beta))$$

Therefore, π_1 is a generalized module homomorphism. Finally, we show that π_1 is surjective. For any $\alpha \in M_1$, since $M_2 \neq \emptyset$, we can choose an arbitrary element $\beta \in M_2$ to construct a pair $(\alpha, \beta) \in M_1 \oplus M_2$. It clearly follows that $\pi_1(\alpha, \beta) = \alpha$. Thus, π_1 is surjective. By Definition 2.5, π_1 is an epimorphism.

By applying an analogous argument, it can be proven that π_2 is also an epimorphism. ■

By utilizing the structural properties established in Theorem 3.9, we can now construct a natural projection mapping from the external direct sum onto its submodule of local identities and examine its homomorphic nature.

Theorem 3.11 Let $M_1 \oplus M_2$ be the external direct sum of generalized modules over a ring R . If M_1 and M_2 are normal generalized modules over a ring R , and $E_1 \oplus E_2$ be its corresponding submodule of local identities. then the mapping $i : M_1 \oplus M_2 \rightarrow E_1 \oplus E_2$ defined by

$$i(\alpha, \beta) = (e_{M_1}(\alpha), e_{M_2}(\beta))$$

for all $(\alpha, \beta) \in M_1 \oplus M_2$ is a generalized module epimorphism.

PROOF. Let $(\alpha, \beta), (\theta, \gamma) \in M_1 \oplus M_2$ such that $(\alpha, \beta) = (\theta, \gamma)$. By the definition of equality in Cartesian products, it follows that $\alpha = \theta$ and $\beta = \gamma$. Since the local identity element for any element in a generalized module is uniquely determined, we have $e_{M_1}(\alpha) = e_{M_1}(\theta)$ and $e_{M_2}(\beta) = e_{M_2}(\gamma)$. Thus $(e_{M_1}(\alpha), e_{M_2}(\beta)) = (e_{M_1}(\theta), e_{M_2}(\gamma))$ which implies $i(\alpha, \beta) = i(\theta, \gamma)$. Hence, the mapping i is well-defined.

For any $(\alpha, \beta), (\theta, \gamma) \in M_1 \oplus M_2$, by applying the fundamental property of local identities under addition, Based Theorem 3.6 we obtain

$$\begin{aligned} i((\alpha, \beta) + (\theta, \gamma)) &= e_{M_1 \oplus M_2}((\alpha, \beta) + (\theta, \gamma)) \\ &= e_{M_1 \oplus M_2}(\alpha, \beta) + e_{M_1 \oplus M_2}(\theta, \gamma) \\ &= i(\alpha, \beta) + i(\theta, \gamma) \end{aligned}$$

Furthermore, evaluating the scalar action for any $r \in R$, we have

$$\begin{aligned} ri(\alpha, \beta) &= r(e_{M_1}(\alpha), e_{M_2}(\beta)) \\ &= (r \cdot_{M_1} e_{M_1}(\alpha), r \cdot_{M_2} e_{M_2}(\beta)) \\ &= (e_{M_1}(r \cdot_{M_1} \alpha), e_{M_2}(r \cdot_{M_2} \beta)) \\ &= i(r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) \\ &= i(r(\alpha, \beta)) \end{aligned}$$

Lastly, for all $(e_{M_1}(\alpha), e_{M_2}(\beta)) \in E_1 \oplus E_2$ there exists an element $(\alpha, \beta) \in M_1 \oplus M_2$ which satisfies $i(\alpha, \beta) = (e_{M_1}(\alpha), e_{M_2}(\beta))$. Therefore, by Definition 2.5, i is a generalized module epimorphism. ■

By utilizing the individual component homomorphisms, we can construct an induced mapping on the external direct sums and verify its algebraic consistency as follows.

Theorem 3.12 let $M_1 \oplus M_2$ and $N_1 \oplus N_2$ be external direct sums of generalized modules. If $f_1 : M_1 \rightarrow N_1$ and $f_2 : M_2 \rightarrow N_2$ are generalized module homomorphisms, then there exists a well-defined mapping $f : M_1 \oplus M_2 \rightarrow N_1 \oplus N_2$ defined by $f(\alpha, \beta) = (f_1(\alpha), f_2(\beta))$ for all $(\alpha, \beta) \in M_1 \oplus M_2$, which is also a generalized module homomorphism.

PROOF. Let $(\alpha, \beta), (\theta, \gamma) \in M_1 \oplus M_2$ such that $(\alpha, \beta) = (\theta, \gamma)$. It follows that $\alpha = \theta$ and $\beta = \gamma$. Based on Definition 2.3, since f_1 and f_2 are well-defined functions, we obtain:

$$f(\alpha, \beta) = (f_1(\alpha), f_2(\beta)) = (f_1(\theta), f_2(\gamma)) = f(\theta, \gamma)$$

Thus, f is well defined. Next for all $(\alpha, \beta), (\theta, \gamma) \in M_1 \oplus M_2$ by applying Definition 2.4 we have

$$\begin{aligned} f((\alpha, \beta) + (\theta, \gamma)) &= f(\alpha +_{M_1} \theta, \beta +_{M_2} \gamma) \\ &= (f_1(\alpha +_{M_1} \theta), f_2(\beta +_{M_2} \gamma)) \\ &= (f_1(\alpha) +_{N_1} f_1(\theta), f_2(\beta) +_{N_2} f_2(\gamma)) \\ &= (f_1(\alpha), f_2(\beta)) + (f_1(\theta), f_2(\gamma)) \\ &= f(\alpha, \beta) + f(\theta, \gamma) \end{aligned}$$

Furthermore, for any $r \in R$ and $(\alpha, \beta) \in M_1 \oplus M_2$ we have

$$rf(\alpha, \beta) = r(f_1(\alpha), f_2(\beta))$$

$$\begin{aligned}
&= (r \cdot_{N_1} f_1(\alpha), r \cdot_{N_2} f_2(\beta)) \\
&= (f_1(r \cdot_{M_1} \alpha), f_2(r \cdot_{M_2} \beta)) \\
&= f(r \cdot_{M_1} \alpha, r \cdot_{M_2} \beta) \\
&= f(r(\alpha, \beta))
\end{aligned}$$

So f is a generalized module homomorphism. ■

Furthermore, the connection between generalized module homomorphisms and the external direct sum construction is given by the following theorem.

Theorem 3.13 If $f(\alpha, \beta) = (f_1(\alpha), f_2(\beta))$ with $f_1 : M_1 \rightarrow N_1$, $f_2 : M_2 \rightarrow N_2$ and f_1 and f_2 are generalized module homomorphisms, then for each $(\alpha, \beta) \in M_1 \oplus M_2$:

1. $e_{N_1 \oplus N_2}(f(\alpha, \beta)) = f(e_{N_1}(\alpha), e_{N_2}(\beta))$
2. $-f(\alpha, \beta) = f(-\alpha, -\beta)$

PROOF. (1) Let $(\alpha, \beta) \in M_1 \oplus M_2$ then $f(\alpha, \beta) = (f_1(\alpha), f_2(\beta)) \in N_1 \oplus N_2$. By the definition of the generalized identity on the external direct sum of the codomain, we have:

$$\begin{aligned}
e_{N_1 \oplus N_2}(f(\alpha, \beta)) &= e_{N_1 \oplus N_2}(f_1(\alpha), f_2(\beta)) \\
&= (e_{N_1}(f_1(\alpha)), e_{N_2}(f_2(\beta)))
\end{aligned}$$

Since f_1 and f_2 are generalized module homomorphisms, they preserve the local identities of their respective components, meaning $e_{N_1}(f_1(\alpha)) = f_1(e_{M_1}(\alpha))$ and $e_{N_2}(f_2(\beta)) = f_2(e_{M_2}(\beta))$. Substituting these into the equation yields:

$$e_{N_1 \oplus N_2}(f(\alpha, \beta)) = (f_1(e_{M_1}(\alpha)), f_2(e_{M_2}(\beta))) = f(e_{N_1}(\alpha), e_{N_2}(\beta))$$

(2) Using Theorem 3.12, Theorem 3.3, And the elementary properties of generalized modules for all $(\alpha, \beta) \in M_1 \oplus M_2$ we have:

$$-f(\alpha, \beta) = -(f_1(\alpha), f_2(\beta)) = (-f_1(\alpha), -f_2(\beta)) = (f_1(-\alpha), f_2(-\beta)) = f(-\alpha, -\beta) \quad \blacksquare$$

4. Conclusion

This paper has successfully investigated the algebraic structure of the external direct sum of generalized modules, establishing several foundational and elementary properties. The main results demonstrate that the external direct sum inherently preserves the essential axiomatic characteristics of generalized modules, while also allowing the formal construction of corresponding submodule structures within this environment. In the context of morphic mappings, this research successfully establishes new generalized module homomorphisms, specifically by constructing canonical projections and inducing combined mappings from individual components onto both standard generalized modules and paired external direct sums. Despite these contributions, this research possesses certain limitations, as the construction and proofs detailed in this study are strictly restricted to the finite case involving only two generalized modules. Furthermore, several preservation properties and epimorphic mappings remain highly dependent on strict constraints, such as the assumption of normality in the underlying structures and the requirement of a ring with unity. For future research, we recommend expanding this framework in three potential directions: investigating the generalization of this structure into full external direct sums involving arbitrary indexed families of generalized modules, exploring the construction of generalized factor modules, and

extending these concepts into the framework of generalized vector spaces to bridge the gap between abstract algebra and generalized linear structures.

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