



RIEMANN–LIOUVILLE FRACTIONAL OPERATORS OF LOGARITHMIC FUNCTION WITH CONVERGENCE ANALYSIS

SYIFAU JANAN^{1,*}, ANDRO KURNIAWAN², DIDI HARLIANTO¹, HILMANA RADHIA PUTERA³, AND AYU MIKA SHERILA⁴

¹Department of Mechanical Engineering, Faculty of Engineering, Universitas Pembangunan Nasional Veteran Jakarta, Depok, Indonesia

²Department of Mathematics, Faculty of Information Technology, Institut Teknologi Batam, Batam, Indonesia

³Department of Industrial Engineering, Faculty of Engineering, Universitas Pembangunan Nasional Veteran Jakarta, Depok, Indonesia

⁴Department of Electrical Engineering, Faculty of Engineering, Universitas Pembangunan Nasional Veteran Jakarta, Depok, Indonesia

*Corresponding Author: syifaul.janan@upnvj.ac.id

ABSTRACT

Fractional calculus extends classical calculus to non-integer orders, yet the systematic study of fractional operators applied to logarithmic functions remains limited. This study derives and analyzes Riemann-Liouville fractional operators applied to $\ln x$ for orders $0 < \alpha \leq 1$. Using Taylor series expansion around $x = 1$, explicit infinite series formulations involving gamma function ratios and fractional powers of $(x - 1)$ are obtained. Rigorous convergence analysis using the ratio test and boundary point examination reveals an important asymmetry: the fractional integral converges on $(0, 2]$, whereas the fractional derivative converges on $(0, 2)$. This difference arises from distinct growth behaviors of the associated gamma functions. Matlab simulations validate theoretical results and demonstrate limiting behaviors: as $\alpha \rightarrow 0$, both operators converge to $\ln x$; as $\alpha \rightarrow 1$, they converge to $x \ln x - x$ and $1/x$ respectively, confirming consistency with classical calculus.

Keywords: *Convergence analysis, Fractional calculus, Logarithmic function, Riemann-Liouville*

1. Introduction

Fractional calculus extends the classical concepts of differentiation and integration to non-integer orders and has developed into a mature mathematical framework with broad theoretical and applied relevance. Since its early conceptual origin in the correspondence between L'Hôpital and Leibniz in 1695 [1], fractional calculus has attracted sustained interest, leading to various definitions and formulations proposed by mathematicians such as Riemann, Liouville, Grünwald, Letnikov, and Caputo [2,3]. Among the existing definitions, the Riemann-Liouville approach plays a fundamental role in the theoretical development of fractional calculus, particularly due to its close connection with integral transforms and series representations [4,5]. In contrast, the Caputo definition is often preferred in physical applications because it preserves classical initial conditions [6], while comparative studies have examined the similarities and differences between these formulations [7,8].

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The theoretical foundations of fractional calculus rely heavily on special functions and convergence analysis of infinite series representations. Recent advances have focused on developing robust mathematical frameworks for fractional operators, including generalized Taylor series expansions [9,10], convolution-based formulations [11], and rigorous convergence criteria [12,13]. The development of efficient numerical schemes for fractional operators has been a major focus, with particular attention to error analysis and stability properties [14,15]. These theoretical developments have established robust frameworks for analyzing fractional operators applied to various classes of functions, emphasizing the role of gamma function ratios in determining convergence properties [16,17].

Most analytical studies in fractional calculus have focused on polynomial, exponential, and trigonometric functions, owing to their well-behaved series expansions and straightforward analytical treatment. However, the natural logarithm function presents intrinsic analytical challenges that have received comparatively limited attention in the fractional calculus literature. Unlike functions that admit a Maclaurin expansion, the logarithmic function is undefined at the origin, making Taylor expansion around a nonzero point necessary [18]. This structural difference limits the direct application of standard fractional calculus techniques. While power functions and their fractional operators have been thoroughly studied [3], with explicit formulas involving gamma function ratios well established, the extension to logarithmic functions requires careful treatment of convergence domains and boundary behavior.

A significant gap exists in the systematic study of Riemann-Liouville fractional operators applied to logarithmic functions. While comprehensive references establish the general framework for fractional calculus [2,3], specific analytical treatment of $\ln x$ with explicit series representations and rigorous convergence analysis remains limited. Recent work on fractional Taylor series and variable-order derivatives has highlighted the importance of careful mathematical treatment [19], yet fundamental questions regarding the precise convergence domains, boundary behavior at critical points, and the asymmetry between convergence intervals of fractional integrals and derivatives for logarithmic functions remain unexplored. The role of gamma function growth rates in determining these convergence properties has not been systematically investigated for this class of functions.

The Riemann-Liouville fractional calculus is especially suitable for addressing these challenges, as it allows fractional operators to be applied term-by-term to infinite series representations. By expressing the natural logarithm as a Taylor series expansion around $x = 1$, fractional integrals and derivatives can be constructed through the fractional calculus of power functions, which are well understood via gamma and beta function relationships. This approach enables the derivation of explicit infinite series representations while preserving mathematical rigor. The convergence analysis of such series requires careful examination of gamma function ratios and their asymptotic behavior, techniques that have been developed in recent studies of fractional operators and series expansions [20].

Motivated by these considerations, this study focuses on the Riemann-Liouville fractional integral and derivative of the natural logarithm function for orders $0 < \alpha \leq 1$. Explicit series representations are derived using Taylor expansion and fractional calculus of power functions, followed by a rigorous convergence analysis employing the ratio test and boundary point examination. The precise convergence intervals are established, revealing an important asymmetry: the fractional integral converges on $(0, 2]$ while the fractional derivative converges on $(0, 2)$, with this difference arising from distinct growth behaviors of the associated gamma functions. Furthermore, numerical simulations using MATLAB are presented to validate the theoretical results and to illustrate the limiting behavior of the fractional operators as the order approaches integer values. Through this analysis, the study clarifies the mathematical structure

of fractional logarithmic operators and contributes to the broader understanding of fractional calculus applied to transcendental functions.

2. Preliminaries

This section presents the fundamental definitions and theoretical results required for the subsequent analysis of fractional calculus applied to the logarithmic function.

Definition 2.1 [2] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f , defined on the interval $[0, x]$, is given by:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\alpha)$ denotes the gamma function. This operator serves as the foundation for the definition of the corresponding fractional derivative.

Definition 2.2 [2] The Riemann-Liouville fractional derivative $f(x)$ of order $n-1 \leq \alpha < n$ is defined as:

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(s)}{(x-s)^{\alpha-n+1}} ds$$

where $\Gamma(n-\alpha)$ denotes the gamma function. Within the scope of this study, with $0 < \alpha \leq 1$, we have $n = 1$, so the fractional derivative definition becomes:

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(s)}{(x-s)^\alpha} ds$$

Theorem 2.3 [18] Let f be a function that is infinitely differentiable in a neighborhood of $x = a$. Then the Taylor series expansion of f about the point a is given by:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

The special case $a = 0$ is known as the Maclaurin series.

Theorem 2.4 [18] Let the power series

$$\sum_{k=0}^{\infty} a_k x^k$$

have radius of convergence $R > 0$, and define:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then f is continuous and differentiable at all points in the interval $|x-a| < R$, and both term-by-term differentiation and integration preserve the same radius of convergence.

3. Fractional Calculus of Power Function

Before analyzing the fractional calculus of the logarithmic function, it is necessary to establish the corresponding results for power function. This step is essential because the logarithmic function will be represented as an infinite series composed of power terms.

Theorem 3.1 Let $\alpha > 0$ and $\beta > -1$. The Riemann-Liouville fractional integral of order α of the power function $f(x) = x^\beta$ is given by

$$I^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}.$$

PROOF. By Definition 2.1, the Riemann–Liouville fractional integral of x^β is

$$I^\alpha x^\beta = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\beta dt.$$

Using the substitution $t = xu$, with $dt = x du$, the integral becomes

$$I^\alpha x^\beta = \frac{x^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^\beta du.$$

The remaining integral corresponds to the beta function $B(\alpha, \beta + 1)$. Applying the beta–gamma relationship,

$$B(\alpha, \beta + 1) = \frac{\Gamma(\alpha)\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)},$$

yields the desired result:

$$I^\alpha x^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^{\beta+\alpha}.$$

■

Theorem 3.2 *Let $0 < \alpha < 1$ and $\beta > -1$. The Riemann–Liouville fractional derivative of order α of the power function $f(x) = x^\beta$ is given by*

$$D^\alpha x^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} x^{\beta-\alpha}.$$

PROOF. For $0 < \alpha < 1$, the Riemann–Liouville fractional derivative is defined as

$$D^\alpha f(x) = \frac{d}{dx} (I^{1-\alpha} f)(x).$$

Applying Theorem 3.1 with order $1 - \alpha$, we obtain

$$I^{1-\alpha} x^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 2 - \alpha)} x^{\beta+1-\alpha}.$$

Differentiating with respect to x yields

$$D^\alpha x^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 2 - \alpha)} (\beta + 1 - \alpha) x^{\beta-\alpha}.$$

Using the gamma function identity

$$\Gamma(\beta + 2 - \alpha) = (\beta + 1 - \alpha)\Gamma(\beta + 1 - \alpha),$$

the expression simplifies to

$$D^\alpha x^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)} x^{\beta-\alpha}.$$

■

4. Fractional Calculus of Logarithmic Function

Having established the Riemann–Liouville fractional calculus for power functions, we now apply these results to the natural logarithm function. Since the logarithmic function is not analytic at the origin, its Taylor expansion about a nonzero point is used as the analytical foundation.

4.1. Fractional Integral of Logarithmic Function

The natural logarithm admits the Taylor series expansion about $x = 1$,

$$I^\alpha \ln x = I^\alpha \left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} (x-1)^k \right), \quad |x-1| < 1$$

Applying the Riemann–Liouville fractional integral operator of order $\alpha > 0$ term-by-term yields

$$I^\alpha \ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} I^\alpha (x-1)^k$$

Using Theorem 3.1, each term can be evaluated explicitly, resulting in

$$I^\alpha \ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} (x-1)^{\alpha+k}$$

To examine the convergence of this series, let

$$a_k = (-1)^{k+1} \frac{\Gamma(k+1)}{k \Gamma(k+\alpha+1)} (x-1)^{k+\alpha}$$

Applying the ratio test gives

$$\left| \frac{a_{k+1}}{a_k} \right| = |x-1| \cdot \frac{k}{k+1} \cdot \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+2)}.$$

Using the recursive property of the gamma function, it follows that

$$\left| \frac{a_{k+1}}{a_k} \right| \sim |x-1| \quad \text{as } k \rightarrow \infty.$$

Therefore, the radius of convergence is $R = 1$, and convergence holds for $|x-1| < 1$.

At the boundary point $x = 0$, the logarithmic function is undefined and therefore excluded from the domain. At $x = 2$, the series becomes alternating with terms of the form

$$\frac{\Gamma(k+1)}{k \Gamma(k+\alpha+1)}.$$

For $\alpha > 0$, the asymptotic relation

$$\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \sim k^{-\alpha} \quad \text{as } k \rightarrow \infty$$

implies that the terms decrease monotonically to zero. Consequently, the series converges at $x = 2$ by the alternating series test. The convergence interval of the fractional integral is therefore $0 < x \leq 2$. Having established the convergence properties of the fractional integral, we now turn to the fractional derivative.

4.2. Fractional Derivative of Logarithmic Function

For $0 < \alpha < 1$, the Riemann–Liouville fractional derivative is applied to the series representation of the logarithmic function as

$$D^\alpha \ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} D^\alpha (x-1)^k.$$

Using Theorem 3.2, each term is evaluated explicitly, yielding

$$D^\alpha \ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma(k+1)}{k \Gamma(k-\alpha+1)} (x-1)^{k-\alpha}.$$

Let

$$b_k = (-1)^{k+1} \frac{\Gamma(k + 1)}{k \Gamma(k - \alpha + 1)} (x - 1)^{k-\alpha}.$$

Applying the ratio test, we obtain

$$\left| \frac{b_{k+1}}{b_k} \right| = |x - 1| \cdot \frac{k}{k + 1} \cdot \frac{\Gamma(k - \alpha + 1)}{\Gamma(k - \alpha + 2)}.$$

As $k \rightarrow \infty$, this expression satisfies

$$\left| \frac{b_{k+1}}{b_k} \right| \sim |x - 1|$$

Therefore, the radius of convergence is again $R = 1$, and convergence holds for $|x - 1| < 1$.

At the boundary point $x = 0$, the logarithmic function is undefined. At $x = 2$, the asymptotic behavior

$$\frac{\Gamma(k + 1)}{\Gamma(k - \alpha + 1)} \sim k^\alpha \quad \text{as } k \rightarrow \infty$$

shows that the terms do not tend to zero for $\alpha > 0$. Hence, the series diverges at $x = 2$. The convergence interval of the fractional derivative is therefore $0 < x < 2$.

The difference between the convergence intervals of the Riemann–Liouville fractional integral and derivative of the logarithmic function arises from the growth behavior of the gamma functions in the denominators. The term $\Gamma(k + \alpha + 1)$ in the fractional integral produces stronger decay, allowing convergence at the boundary point $x = 2$, whereas $\Gamma(k - \alpha + 1)$ in the fractional derivative results in weaker decay and leads to divergence at the same point.

5. Numerical Simulation

To validate the theoretical results derived in the previous sections, numerical simulations were conducted using MATLAB. The simulations aim to illustrate the convergence behavior of the Riemann–Liouville fractional integral and fractional derivative of the logarithmic function for various fractional orders $\alpha \in (0,1)$, as well as their limiting behavior as α approaches integer values.

5.1. Fractional Integral of Logarithmic Function

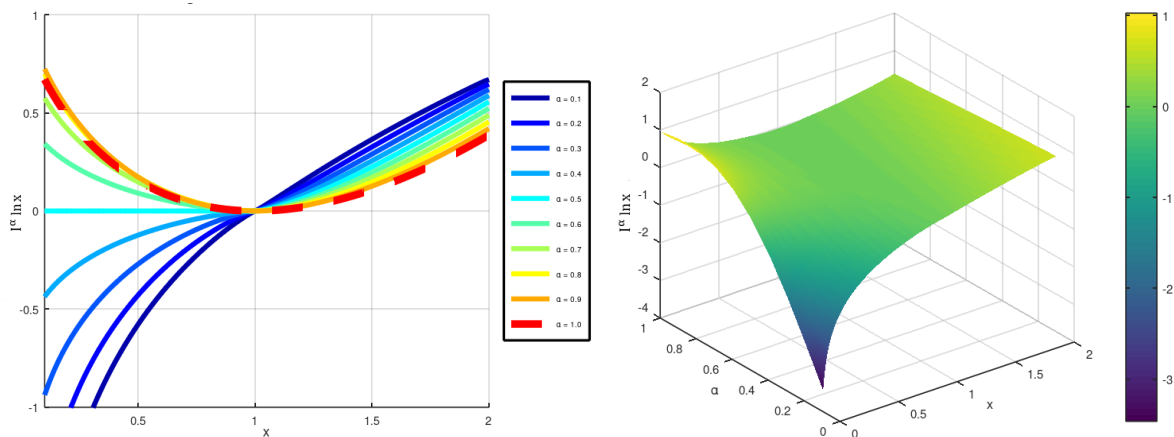


Figure 1. 2D and 3D visualization of fractional integral $I^\alpha \ln x$ for various orders $\alpha \in (0,1]$.

Figure 1 presents two-dimensional and three-dimensional visualizations of the Riemann–Liouville fractional integral $I^\alpha \ln x$ for several values of α over the interval $0 < x \leq 2$. In the

two-dimensional plots, each curve corresponds to a fixed fractional order, illustrating how the fractional integral varies continuously with respect to α .

As $\alpha \rightarrow 0$, the numerical results converge to the natural logarithm function $\ln x$, which is consistent with the identity property of the fractional integral. Conversely, as $\alpha \rightarrow 1$, the results converge to the classical integral $\int \ln x dx$. For intermediate values of α , the curves exhibit smooth transitions between these two limiting cases, confirming the continuous dependence of the fractional integral on the order α .

The three-dimensional surface plot further demonstrates the combined dependence of $I^\alpha \ln x$ on both the spatial variable x and the fractional order α . The smooth surface observed over the domain indicates numerical stability and supports the convergence interval $0 < x \leq 2$ established analytically in Section 4.1.

5.2. Fractional Derivative of Logarithmic Function

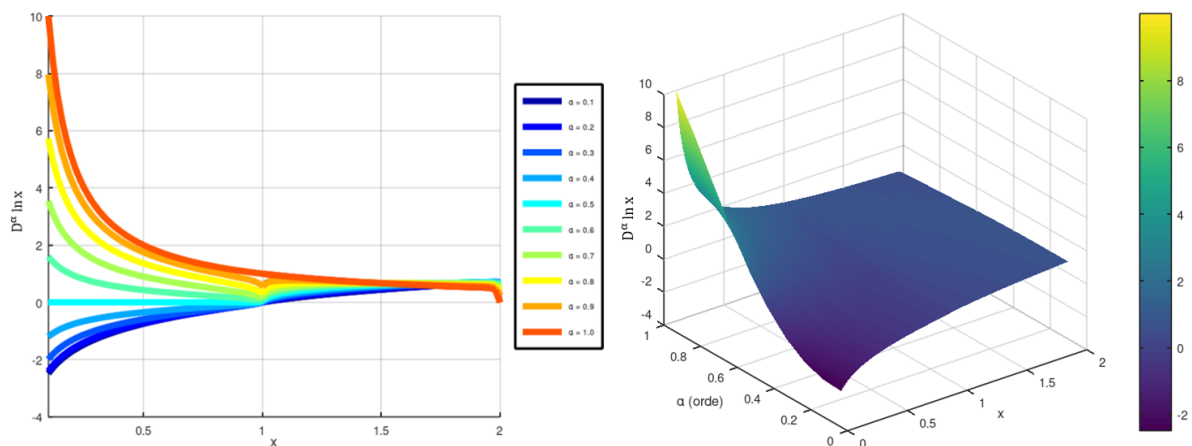


Figure 2. 2D and 3D visualization of fractional derivative $D^\alpha \ln x$ for various orders $\alpha \in (0,1]$.

Figure 2 illustrates the numerical behavior of the Riemann–Liouville fractional derivative $D^\alpha \ln x$ for various values of α on the interval $0 < x < 2$. The two-dimensional plots show that, as $\alpha \rightarrow 0$, the fractional derivative converges to the logarithmic function $\ln x$, while as $\alpha \rightarrow 1$, it converges to the classical first derivative $1/x$.

For intermediate values of α , the numerical curves display characteristic fractional behavior, including increasing sensitivity near the lower boundary of the domain. This behavior is consistent with the analytical result that the fractional derivative does not converge at the boundary point $x = 2$, as established in Section 4.2.

The three-dimensional surface plot highlights the continuous variation of $D^\alpha \ln x$ with respect to both x and α , while clearly excluding the boundary point $x = 2$ due to divergence. These numerical observations confirm the theoretical convergence interval $0 < x < 2$ and illustrate the asymmetry between the fractional integral and fractional derivative.

6. Conclusion

This study has provided comprehensive analytical treatment of Riemann–Liouville fractional operators applied to the natural logarithm function for orders $0 < \alpha \leq 1$. Through Taylor series expansion about $x = 1$ and systematic application of fractional calculus to power functions, explicit infinite series representations have been derived for both fractional integrals and derivatives.

The rigorous convergence analysis, employing the ratio test and boundary point examination, has revealed a fundamental asymmetry in convergence domains: the fractional integral converges on $(0, 2]$, while the fractional derivative converges on $(0, 2)$. This difference, originating from distinct growth behaviors of gamma functions in the series

denominators, represents a key contribution to the theoretical understanding of fractional operators applied to transcendental functions.

Numerical simulations performed using MATLAB further confirm the theoretical findings. The simulations demonstrate smooth transitions between limiting cases as the fractional order varies, with the fractional operators converging to the natural logarithm as $\alpha \rightarrow 0$, and to the corresponding classical integral and derivative as $\alpha \rightarrow 1$. These numerical results provide strong validation of the analytical convergence properties and boundary behavior derived in this study.

Numerical simulations using MATLAB have validated all theoretical findings, demonstrating smooth transitions between limiting cases and confirming the predicted convergence intervals. The observed limiting behaviors convergence to $\ln x$ as $\alpha \rightarrow 0$ and to classical operators as $\alpha \rightarrow 1$ establish consistency with classical calculus principles.

The analytical framework developed here extends beyond the logarithmic function. The methodology combining Taylor series expansion, term-by-term fractional operator application, and gamma function asymptotic analysis can be applied to other transcendental functions such as inverse trigonometric functions, Lambert W function, and generalized logarithms. Future research directions include: (1) extension to higher-order fractional operators ($\alpha > 1$), (2) investigation of Caputo fractional operators for comparison, (3) application to fractional differential equations involving logarithmic nonlinearities, and (4) development of efficient numerical schemes exploiting the series representations derived in this work.

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