J. Ris. & Ap. Mat. Vol. 9 No. 2 (2025) pp. 236-243
Jurnal Riset dan Aplikasi Matematika (JRAM)
e-ISSN: 2581-0154
URL: journal.unesa.ac.id/index.php/jram

CONNECTIVITY OF COPRIME GRAPHS OVER CYCLIC GROUPS

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ABSTRACT

A coprime graph over a finite group is a representation of a finite group on a graph whose vertex set is an element of the group G and two distinct vertices $x, y \in G$ are connected if and only if the orders of the two elements are mutually prime. This research discusses the connectivity of coprime graphs over cyclic groups and their subgroups. Since any cyclic group is isomorphic to the group $\mathbb{Z}_n$ which is the group modulo integers n, the discussion in this article utilizes the group $\mathbb{Z}_n$. In the previous research, has been discussed the patterns that appear on the coprime graph of group $\mathbb{Z}_n$ and its subgroups. Based on these patterns, the connectivity of the coprime graph over the cyclic group related to Euler, Hamilton, and Planar graphs is obtained as well as the diameter and girth of the coprime graph.

Keywords: *Coprime Graph, Euler, Hamiltonian, Planar, Cyclic.*

1 Introduction

Graph theory is an appropriate basis for implementing the properties of a group. Graph representation of groups was first introduced by A. Cayley in 1878 by representing a finite group. The graph represents the group G and A is a subset of G so that the graph formed is a directed graph or digraph and is called a Cayley Graph. The set of vertices in the graph is the set of elements in the group and two vertices x and y are connected if and only if there exist $a \in A$ such that $y = ax$. Since the graph formed is an undirected graph, there is another research by modifying the definition so that the graph formed is an undirected graph. Further research on Cayley Graphs is done by changing the finite group object into a semigroup which is discussed by [1] and [2]. Cayley graph in directed graph version is discussed in [3] and [4]. While the discussion by changing the object to a semigroup is found in research [5]. Cayley graph results on a specific group, namely the generalized quaternion group, are discussed in research [6].

Research [7] introduced various kinds of graphs defined on semigroups. While in research [1] introduced rank graphs on semigroups S as directed graphs whose vertex set is an element in S and two distinct elements $a,b \in S$ are connected if and only if $b = am$ for a positive integer m. Furthermore, [8] introduced a new graph known as the order super graph of the rank graph

2020 *Mathematics Subject Classification*: 05C40, 05C25
Diterima: 12-08-25; direvisi: 10-10-25; diterima: 28-10-25

over the finite group G whose definition is related to the power graph, enchanted power graph and commuting graph. The automorphism group of this graph is also investigated in [9].

Representations of groups on graphs involving the order of the group get a lot of attention from mathematical researchers, one of which is the element order graph that appears in the study [10]. Another representation of groups on graphs involving group orders is the coprime graph. The definition of a coprime graph of a finite group G is formed by viewing all elements in the group as vertices, and elements x, y are connected if the order of the two elements is prime in [8]. Research on coprime graph was further conducted by [11] by studying its various properties. Coprime graphs are studied extensively in [12] looking at the connectivity index of coprime graphs over certain groups, while [13] by looking specifically at its properties on generalized quaternion groups. A modification of this coprime graph is the coprime prime graph, by changing the definition of connectedness by adding the condition that two vertices are connected if their greatest common factor is prime. One of the researches that discuss coprime graphs is [14].

Research [15] discusses the patterns that appear on the coprime graph of the integer group modulo n and its subgroups. Based on this research, researchers want to continue this research by studying the connectivity on the coprime graph which has never been discussed in any research. Therefore, the researcher discusses the connectivity of the coprime graph over the cyclic group which in this case is represented as an integer group modulo n and its subgroups including Euler, Hamilton, and Planar as well as the diameter and girth of the coprime graph.

2 Basic Theory

The following are some of the definitions used by researchers, including definitions in group theory and graph theory.

Definition 2.1[16] The set $\mathbb{Z}_n = \{0,1,\cdots,n-1\}$ with $n \geq 1$ is a group over the addition operation modulo n. For every $i \in \mathbb{Z}_n$ the inverse of i is $n - i$. The group $\mathbb{Z}_n$ is commonly called the group of integers modulo n.

Definition 2.2[4] Given $H \neq \emptyset$ with $H \subseteq G$. The set H is a subgroup of G if H is a group with respect to the same operation on G, and its denoted by $H \leq G$.

Definition 2.3[4] Given H a subgroup of G. A subgroup H is said to be a non-trivial subgroup if there exists another element in the group G which is neither $\{e\}$ nor G itself.

Definition 2.4[4] Given G is a group. The order of an element of $g \in G$ is the smallest natural number$n$ that satisfies $gn = e$. If no natural number n satisfies then the order g is called infinite order. The order of an element of $g \in G$ is n denoted by $o(g) = n$.

Example 2.5 Given the group $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ is an addition group modulo 5. Apart from the identity element which has order 1, all other elements have order 5.

Definition 2.6[17] Given a graph G whose vertex set can be partitioned into two subsets V_1 and V_2 such that every edge in G connects a vertex in V_1 to a vertex in V_2 is said to be a bipartite graph denoted $G(V_1, V_2)$. If every vertex in V_1 is connected to all vertices in V_2 then $G(V_1, V_2)$ is called a complete bipartite graph denoted by $K_{m,n}$ with $m = |V_1|$ and $n = |V_2|$.

Definition 2.7[17] A k-partite graph is a graph whose vertices can be partitioned into k unordered sets such that no two vertices in the set are connected.

Definition 2.8[17] A graph is an Eulerian graph if and only if each vertex has even degree.

Definition 2.9[17] A graph G is a hamilton graph if G has a vertex that passes through all vertices in the graph exactly once.

Definition 2.10[17] Graph G is a planar graph if and only if G contains no subgraph of $K_{3,3}$ or K_5.

Definition 2.11[18] The girth of a G graph is the minimum length of a vertex contained in the G is denoted by $g(G)$. If there is no cycle formed in the graph G then $g(G) = \infty$.

Definition 2.12[17] Given x and y distinct vertices in the graph G. The distance from x to y is the length of the shortest path in the graph G from vertex x to vertex y denoted by $d(x, y)$. If there is no path between vertex x to y then $d(x, y) = \infty$.

Definition 2.13[18] Given a connected multigraph G with $v \in V(G)$. The eccentricity of G is the distance from v to the farthest vertex in G denoted by $e(v)$.

Definition 2.14[19] The diameter of a graph G is the maximum eccentricity at each vertex in G denoted by $diam(G)$.

3 Result and Discussion

The discussion of coprime graphs on integer groups modulo$n$ and their subgroups begins by revisiting the definition of coprime graphs on undirected graphs. The following definition is given more clearly.

Definition 3.1[8] The coprime graph of a group G is a graph whose vertices are elements of G and two distinct vertices u and$v$ are connected if and only if $\gcd(o(u), o(v) = 1$ is denoted by $\Gamma(G)$.

Based on this definition, we will look for patterns that arise from coprime graphs over the group of integers modulo n. The following example is given to illustrate the coprime graph over group $\mathbb{Z}_n$.

Example 3.2 Let $\mathbb{Z}_n$ be a cyclic group with $\mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\}$. We obtain the coprime graph over $\mathbb{Z}_{13}$ as follows

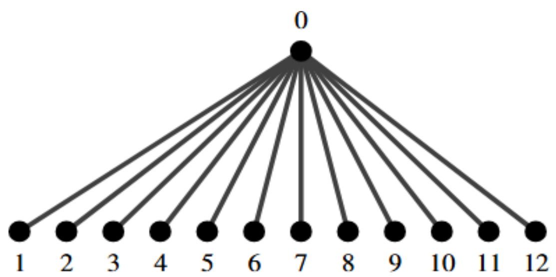

Figure 1: Coprime Graph over $\mathbb{Z}_{13}$

By looking at the example above, the properties of the coprime graph over group $\mathbb{Z}_n$ are obtained as follows. The first result obtained from the coprime graph over $\mathbb{Z}_n$ is a complete bipartite graph for every n is a prime number.

Theorem 3.3[15] If n is prime then the coprime graph over $\mathbb{Z}_n$ is a complete bipartite graph.

Example 3.4 Given the group Z_{25}. The coprime graph form over Z_{25} is obtained as follows

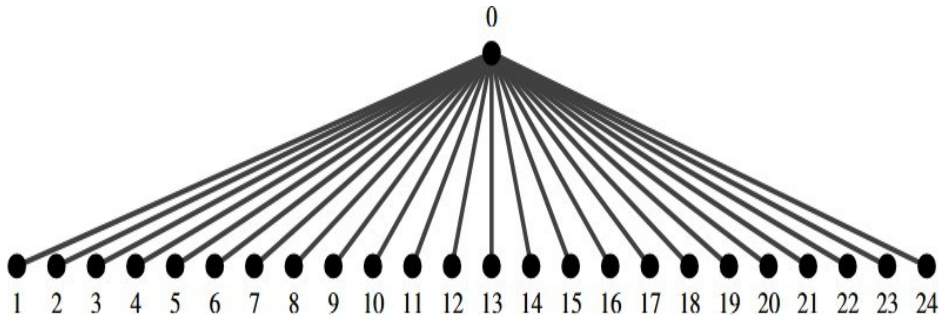

Figure 2: The coprime graph $\mathbb{Z}_{25}$

Based on the above, it is also obtained that the coprime graph over $\mathbb{Z}_n$ is a complete bipartite graph if n is power of prime.

Theorem 3.5[15] If $n = p^k$ with p is prime and $k \in \mathbb{N}$ then the coprime graph over $\mathbb{Z}_n$ is a complete bipartite graph.

Example 3.6 Given the group $\mathbb{Z}_{30}$. By considering the order of the elements of each member of the group, the coprime graph of $\mathbb{Z}_{30}$ is obtained as follows

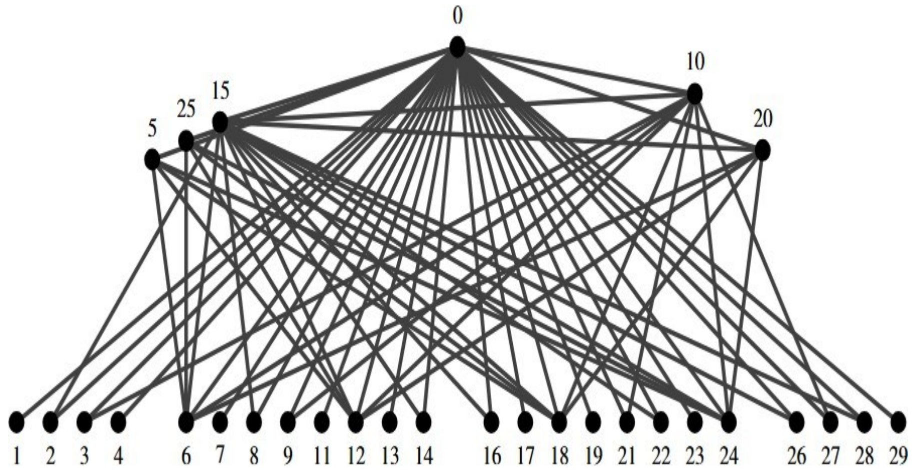

Figure 3: The coprime graph $\mathbb{Z}_{30}$

The next result is given that the coprime graph over $\mathbb{Z}_n$ with$n$ not a prime power is a t-partite graph.

Theorem 3.7[15] If$n = p_1^{k_1} p_2^{k_2} \dots p_j^{k_j}$ with $p_1, p_2, \dots, p_j$ are distinct primes and $k_1, k_2, \dots, k_j$ is a natural number then the coprime graph over $\mathbb{Z}_n$ is a $(j + 1) -$ partite graph.

Next, the connectivity of the resulting graph will be investigated. The first part to be discussed relates to the Eulerian of the resulting graph. As a result of always finding vertices with degree one, any coprime graph over group $\mathbb{Z}_n$ is not an Eulerian or Hamiltonian graph. The following theorem explains this.

Theorem 3.8 Given the group $\mathbb{Z}_n$. For every n then the graph $\Gamma(\mathbb{Z}_n)$ is not an euler graph.

PROOF. Proof is divided into several cases:

1. For $n = p$. Based on Theorem 3.3, it is obtained that the set $\mathbb{Z}_n$ can be partitioned into two sets $V_1 = \{0\}$ and $V_2 = \{1, 2, \dots, n - 1\}$. Therefore, for every $x, y \in V_2$ is not connected and only connected with element 0. This means that $deg(x) = deg(y) = 1$ so that every

vertex other than 0 will have odd degree. Thus, it is obtained that the coprime graph over $\mathbb{Z}_n$ is not an euler graph.

2. For $n = p^k$. Evidence in line with $n = p$.
3. For $n = p_1^{k_1} p_2^{k_2} \dots p_j^{k_j}$. It is known that $x, y \in \mathbb{Z}_n$, since $o(x) = o(y) = n = p_1^{k_1} p_2^{k_2} \dots p_j^{k_j}$, then the order for each element is not coprime with the order of element x. Therefore, vertices x and y are only connected to element 0. This means that $deg(x) = deg(y) = 1$. Thus, it is obtained that the coprime graph over $\mathbb{Z}_n$ is not an euler graph.

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Theorem 3.9 Given the group $\mathbb{Z}_n$. For every n then the graph $\Gamma(\mathbb{Z}_n)$ is not a Hamilton graph.
PROOF. Proof is divided into several cases:

1. For $n=p$. It is known that for every $x, y \in \mathbb{Z}_n$ with $x, y \neq 0$ then $\gcd(o(x), o(y)) \neq 1$. Therefore, for every $x, y \in \mathbb{Z}_n$ is not connected and only connected with element 0. Thus, it is found that there is no cycle formed and hence the coprime graph of $\mathbb{Z}_n$ is not a Hamiltonian graph.
2. For $n=p^k$. Evidence in line with $n=p$.
3. For$n = p_1^{k_1} p_2^{k_2} \dots p_j^{k_j}$. It is known that for every $a \in \mathbb{Z}_n$ with $o(a) \neq 1$ then $o(a) = p_1^{k_1-l_1} p_2^{k_2-l_2} \dots p_j^{k_j-l_j}$. Furthermore, for every $b \in \mathbb{Z}_n$ with $o(b) \neq 1$ it follows that $o(b) = p_1^{k_1} p_2^{k_2} \dots p_j^{k_j}$. Since $o(b) = n$ then the vertex b will only have one edge i.e. the edge connecting b with 0. Consequently, there will be no cycle formed and passing through the vertex b. Thus, it is obtained that the coprime graph over $\mathbb{Z}_n$ is not a hamilton graph.

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For $n = p$ and $n = p^k$ where p is prime, the coprime graph formed from $\mathbb{Z}_n$ is a star graph. This guarantees the graph to be a planar graph, as appears in the following theorem.

Theorem 3.10 Given the group $\mathbb{Z}_n$. For $n = p$ or $n = p^k$ where p is prime, the graph $\Gamma(\mathbb{Z}_n)$ is a planar graph.
PROOF. Given $n = p$ or p^k with p is a prime number. According to Theorem 3.3 and Theorem 3.4 the coprime graph over $\mathbb{Z}_n$ is a complete bipartite graph. More precisely, the coprime graph formed by is a *star graph*. Thus, it is obtained that the coprime graph over $\mathbb{Z}_n$ is a planar graph.

Planarity in the coprime graph over group Z_n does not occur for any n. For example, consider the coprime graph formed over group Z_{30} in Example 3.6. Consider the subgraph formed from the element $S = \{5,15,25,6,12,18\}$. If S is partitioned into $V_1 = \{5,15,25\}$ and $V_2 = \{6,12,18\}$ then a bipartite graph $K_{3,3}$ is formed. This shows that the coprime graph over Z_{30} contains the subgraph $K_{3,3}$, meaning that it is not a planar graph. Thus, the planarity guarantee of the formed graph can only be found for $n = p$ and $n = p^k$ where p is a prime number.

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Since any vertex in the coprime group over $\mathbb{Z}_n$ is always connected with the identity element, the most dependent path connecting any two vertices is 2. This gives rise to the following theorem.

Theorem 3.11 Given a group $\mathbb{Z}_n$ then $diam\ \Gamma(Z_n) \leq 2$.
PROOF. Proof is divided into several cases:

1. For $n > 2$ then $diam\ \Gamma(Z_n) = 2$. Given $\mathbb{Z}_n$ the group of integers modulo n. Given that

every vertex is directly connected to 0, the distance of any two distinct vertices in $\Gamma_{\mathbb{Z}_n}$ contains only two possibilities, namely $1 \leq d\ (u, v) \leq 2$ for every $u, v \in V(\Gamma(\mathbb{Z}_n)$. Two distinct vertices are 1 apart if they are directly connected and 2 apart if they are not directly connected. Since diameter is the largest eccentricity, we get $diam(\Gamma(\mathbb{Z}_n)) = 2$.

2. For $n = 2$ then $diam\ \Gamma(\mathbb{Z}_n) = 1$. Given $\mathbb{Z}_n$ is an integer group modulo n for $n = 2$. It is obtained that the coprime graph over the integer group modulo 2 is the complete graph K_2. As a result, every two vertices are connected. Thus, the eccentricity of each vertex in $V(\Gamma(\mathbb{Z}_n))$ is 1. Thus, it is obtained that diam $\Gamma(\mathbb{Z}_n) = 1$.

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As a result of the fact that any vertex is always connected with the identity element, if two verticesx, y (no identity element in them) are connected to each other, they will form a spindle with the identity element. This is the basis of the following theorem.

Theorem 3.12 Given a group $\mathbb{Z}_n$ then $g(\Gamma_{\mathbb{Z}_n})$ is 3 or ∞.

PROOF. Given $\mathbb{Z}_n$ the group of integers modulo n. If there are $x, y \in \mathbb{Z}_n$ with $x, y \neq e$ so that x and y are connected, we get a cycle of order 3, namely $\{x, e, y\}$ which means that we get $g(\Gamma(\mathrm{Z}_n))$= 3. Whereas if $x, y \in \mathbb{Z}_n$ with $x, y \neq e$ so that x and y are not connected to each other then there are no particles formed, meaning that $g(\Gamma(\mathrm{Z}_n))$ = ∞.

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Based on Theorem 3.3, Theorem 3.5, and Theorem 3.7, it is known that the coprime graph on group $\mathbb{Z}_n$ will form a bipartite, complete bipartite, or multipartite graph, respectively. Furthermore, it is known that in the upper subgroup of group $\mathbb{Z}_n$ the coprime graph formed is a bipartite graph or multipartite graph. Thus, the connectivity of the coprime graph in group $\mathbb{Z}_n$ is similar to the connectivity in its subgroups. Thus, it is found that the coprime graph of the upper subgroup of group $\mathbb{Z}_n$ is neither an Euler graph nor a Hamilton graph. However, the coprime graph is a planar graph. In addition, the diameter and *girth of* the coprime graph of the upper subgroup $\mathbb{Z}_n$ are also the same as those of the group. So, the diameter of the subgroup is always less than or equal 2 and its *girth* has two possibilities, either 3 or ∞. The proof of connectivity in the upper subgroup of group $\mathbb{Z}_n$ is in line with the proofs in Theorem 3.8 to Theorem 3.12.

4 Conclusion

Based on the coprime graph formed on group $\mathbb{Z}_n$ and its subgroups for every n. In general, the coprime graph formed on group $\mathbb{Z}_n$ is a bipartite graph for any$n$ prime power while for$n$ non-prime power, it is a multipartite graph. However, for non-trivial subgroups of group $\mathbb{Z}_n$ it is found that the coprime graph can be a bipartite graph even when$n$ is not a prime power.

Furthermore, it is found that for every$n$ in group $\mathbb{Z}_n$, the coprime graph formed is neither an euler graph nor a hamilton graph but a planar graph. In addition, the diameter of group $\mathbb{Z}_n$ will always less than or equal 2 while the *girth* will always be 3 or ∞. These properties of group $\mathbb{Z}_n$ will also apply to the *non-trivial* subgroups of group $\mathbb{Z}_n$.

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