



# FORGOTTEN, HARARY, AND SCHULTZ INDEXES OF PRIME COPRIME GRAPHS OF THE MODULO GROUP OF INTEGER NUMBERS

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## ABSTRAK

Indeks topologi merupakan hal mendasar dalam teori graf karena indeks ini menyediakan ukuran numerik yang menangkap karakteristik struktur dan pola konektivitas suatu graf. Tujuan pada penelitian ini untuk dapat menemukan rumus umum dari indeks Forgotten, indeks Harary, dan indeks Schultz, dari graf koprima prima dari grup bilangan bulat modulo bilangan bulat positif. Graf koprima prima didefinisikan sebagai graf dimana dua sebarang simpul dikatakan bertetangga jika dan hanya jika faktor persekutuan terbesar (FPB) dari kedua orde simpul tersebut bernilai 1 atau prima.

**Kata Kunci:** Graf Koprima Prima, indeks Forgotten, indeks Harary, indeks Schultz

## ABSTRACT

*Topological indices are fundamental to graph theory as they provide numerical measures that capture the structural characteristics and connectivity patterns of a graph. The objective of this study is to find the general formulas for the Forgotten index, Harary index, and Schultz index of the prime coprime graph derived of the group of integers modulo some positive integer. A prime coprime graph is defined as a graph where any two vertices are adjacent if and only if the greatest common divisor (GCD) of the orders of the two vertices is either 1 or a prime number.*

**Keywords:** Prime Coprime Graph, Forgotten Index, Harary Index, Schultz Index

## 1 Introduction

Graph theory is a rapidly developing branch of mathematics with wide applications across various fields such as computer science, chemistry, biology, and network theory. One of the main research objects in graph theory is the topological index, a numerical parameter that represents various structural properties of a graph. Topological indices are used to describe graph characteristics through certain numbers that remain constant under graph automorphisms.

Some well-known topological indices include the Forgotten, Harary, and Schultz indices, each playing an important role in measuring various structural aspects of graphs [1][2][3].

Several studies on topological indices include the first Zagreb index, the Wiener index, and the Gutman index of the power of dihedral group [4], and Harary index of coprime graph of the group of integers modulo of prime order [5], Hyper-Wiener index and Padmakar Ivan index of graph coprime of group dihedral [6]. Graphs that have been studied in this context include prime graphs, coprime graphs, non-coprime graphs, dihedral graphs, and power graphs. Some graphs have only had their basic properties discussed thus far. One particularly interesting type of graph is the prime coprime graph. A prime coprime graph of a group  $G$  is a graph whose vertices consist of all elements of a group  $G$ , where two distinct vertices are *adjacent* if and only if the greatest common divisor (GCD) of their orders is 1 or a prime number [7]. The aim of this study is to obtain general formulas for the Forgotten, Harary, and Schultz indices on coprime prime graphs of the group of integers modulo some positive integer. The aim of this research is to analyze the Forgotten, Harary, and Schultz indices of prime coprime graphs derived from the modulo group of integers, in order to deepen the understanding of graph invariants in modular arithmetic graph structures, as explored in prior studies on graphs representable modulo  $n$  and congruence-based graphs, and to support further developments in graph theory and its practical applications [8].

## 2 Literature Review

Graph theory continues to expand rapidly, supported by research that introduces various types of graphs and studies their structural properties. In this study, a special focus is placed on the prime coprime graph, a structure derived from the group of integers modulo some positive integer, and the associated topological indices. To support a deeper understanding, several basic definitions are introduced below.

Before discussing the structure of prime coprime graphs, it is essential to understand the concept of the group of integers modulo some positive integer.

**Definition 2.1**[9] The group  $\mathbb{Z}_n$  is called the group of integers modulo  $n$ , consisting of the set of all non-negative integers equipped with the addition modulo  $n$  operation, denoted by the set:

$$\mathbb{Z}_n = \{0, 1, 2, \dots, n - 1\}$$

After understanding the basic group structure, we define the concept of a prime coprime graph, which forms the foundation for this research.

**Definition 2.2** [7] Given a finite group  $G$  such that  $|G| > 2$ . The prime coprime graph is defined as the graph  $\Gamma_G = (V, E)$  consists of all vertex of  $G$  and which two distinct vertices  $u$  and  $v$  are adjacent if and only if  $GCD(|x|, |y|) = 1$  or  $GCD(|x|, |y|) = p$ .

The structure of a prime coprime graph for  $n = p^k$ , where  $p$  is a prime number and  $k \geq 2$  is an integer, is based on the set  $\mathbb{Z}_n = \{0, 1, 2, \dots, p^k - 1\}$ . In this graph, two distinct vertices  $u$  and  $v$  are *adjacent* if and only if  $GCD(|u|, |v|) = 1$  or  $GCD(|u|, |v|) = p$ , the vertices not divisible by  $p$  are connected to each other by edges where the  $GCD$  of two vertices is 1.

Meanwhile, vertices that are multiples of  $p$  can also be connected if their **GCD** is  $p$ . This results in a graph with connectivity patterns that depend on the **GCD** of each pair of vertices.

**Example 2.3:** Suppose we are given the group  $G = \mathbb{Z}_8$  the integers modulo 8, here  $n=8$  with  $n = 2^3$  where  $p = 2$  and  $k = 3$  let us write:

$$\mathbb{Z}_8 = \{e, 1, 2, 3, 4, 5, 6, 7\}$$

Then, the prime coprime graph for  $\mathbb{Z}_8$  is depicted on Figure 1:

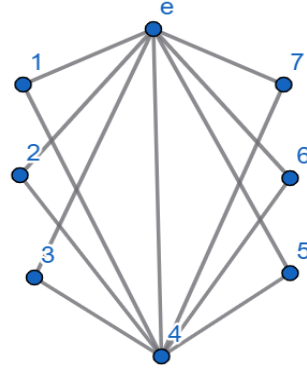


Figure 1: Prime coprime graf of  $\mathbb{Z}_8$

To measure the structure of a prime coprime graph, one of the topological indices used is the Schultz index, defined as follows.

**Definition 2.4** [10] Let  $G$  be a graph. The Schultz index of  $G$  denoted by  $S(G)$ , is defined as:

$$S(G) = \sum_{\{u,v\} \subset V} (\deg(u) + \deg(v)) d(u,v)$$

Where  $\deg(u)$ ,  $\deg(v)$  is the degree of vertex  $v$  in  $G$ , and  $d(u, v)$  is the distance between vertices  $u$  and  $v$  in  $G$ , and the pair  $\{u, v\}$  and  $\{v, u\}$  is only counted once.

**Example 2.5** Suppose the prime coprime graph of the group  $\mathbb{Z}_8$  denoted by  $\Gamma_{\mathbb{Z}_8}$ . Then, the Schultz index is obtained as follows:

$$\begin{aligned} S(\Gamma_{\mathbb{Z}_8}) = & ((\deg(e) + \deg(4)) \cdot d(e, 4)) + ((\deg(e) + \deg(1)) \cdot d(e, 1)) \\ & + ((\deg(e) + \deg(2)) \cdot d(e, 1)) + ((\deg(e) + \deg(3)) \cdot d(e, 1)) \\ & + ((\deg(e) + \deg(5)) \cdot d(e, 1)) + \\ & ((\deg(e) + \deg(6)) \cdot d(e, 1)) + ((\deg(e) + \deg(7)) \cdot d(e, 1)) + \\ & ((\deg(4) + \deg(1)) \cdot d(4, 1)) + ((\deg(4) + \deg(2)) \cdot d(4, 2)) + \\ & ((\deg(4) + \deg(3)) \cdot d(4, 3)) + ((\deg(4) + \deg(5)) \cdot d(4, 5)) + \\ & ((\deg(4) + \deg(6)) \cdot d(4, 6)) + ((\deg(4) + \deg(7)) \cdot d(4, 7)) + \\ & ((\deg(1) + \deg(2)) \cdot d(1, 2)) + ((\deg(1) + \deg(3)) \cdot d(1, 3)) + \\ & ((\deg(1) + \deg(5)) \cdot d(1, 5)) + ((\deg(1) + \deg(6)) \cdot d(1, 6)) + \\ & ((\deg(1) + \deg(7)) \cdot d(1, 7)) + ((\deg(2) + \deg(3)) \cdot d(2, 3)) + \\ & ((\deg(2) + \deg(5)) \cdot d(2, 5)) + ((\deg(2) + \deg(6)) \cdot d(2, 6)) + \\ & ((\deg(2) + \deg(7)) \cdot d(2, 7)) + ((\deg(3) + \deg(5)) \cdot d(3, 5)) + \\ & ((\deg(3) + \deg(6)) \cdot d(3, 6)) + ((\deg(3) + \deg(7)) \cdot d(3, 7)) + \end{aligned}$$

$$\begin{aligned}
& ((\deg(5) + \deg(6) \cdot d(5, 6)) + ((\deg(5) + \deg(7) \cdot d(5, 7)) + \\
& ((\deg(6) + \deg(7) \cdot d(6, 7)) \\
& ((7 + 7) \cdot 1) + ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + \\
& ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + \\
& ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + ((7 + 2) \cdot 1) + \\
& ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + \\
& ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + \\
& ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + \\
& ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) + ((2 + 2) \cdot 2) \\
& (1 \times (7 + 7) \times 1) + (12 \times (7 + 2) \times 1) + (15 \times (2 + 2) \times 2) \\
& = 14 + 108 + 120 \\
& S(\Gamma_{\mathbb{Z}_8}) = 242
\end{aligned}$$

Another important topological index in graph theory is the Forgotten index, which captures different structural properties, as defined below.

**Definition 2.6** [11] Let  $G$  be a graph. The Forgotten index of  $G$  denoted by  $F(G)$ , is defined as:

$$F(G) = \sum_{v \in V} \deg(v)^3$$

Where  $\deg(v)$  is the degree of vertex  $v$ , for all vertex in  $G$ .

The Harary index is another key measure that focuses on the distances between vertices in the graph. Its definition is presented below.

**Example 2.7** Consider again the prime coprime graph of the group  $\mathbb{Z}_8$  denoted by  $\Gamma_{\mathbb{Z}_8}$ . Then, the Forgotten index is obtained as follows:

$$\begin{aligned}
F(\Gamma_{\mathbb{Z}_8}) &= (\deg(e))^3 + \deg(1)^3 + \deg(2)^3 + \deg(3)^3 + \deg(4)^3 + \deg(5)^3 + \deg(6)^3 \\
&\quad + \deg(7)^3 \\
&= (7)^3 + (2)^3 + (2)^3 + (2)^3 + (7)^3 + (2)^3 + (2)^3 + (2)^3 \\
&= 343 + 8 + 8 + 8 + 343 + 8 + 8 + 8 \\
&= (2 \times 343) + (6 \times 8) \\
&= 686 + 48 \\
F(\Gamma_{\mathbb{Z}_8}) &= 734
\end{aligned}$$

**Definition 2.8** [5] Let  $G$  be a graph. The Harary index of  $G$  denoted by  $H(G)$ , is defined as:

$$H(G) = \sum_{\{u,v\} \subset V} \frac{1}{d(u,v)}$$

Where  $d(u, v)$  denotes the shortest path distance between vertices  $u$  and  $v$  in  $G$ , and the pair  $(u, v) = (v, u)$  is only counted once.

**Example 2.9** Now, let us see again the prime coprime graph of the group  $\mathbb{Z}_8$  denoted by  $\Gamma_{\mathbb{Z}_8}$ . Then, the Harary index is obtained as follows:

$$\begin{aligned}
H(\Gamma_{\mathbb{Z}_8}) &= \frac{1}{d(e,1)} + \frac{1}{d(e,2)} + \frac{1}{d(e,3)} + \frac{1}{d(e,4)} + \frac{1}{d(e,5)} + \frac{1}{d(e,6)} + \frac{1}{d(e,7)} + \frac{1}{d(1,2)} \\
&\quad + \frac{1}{d(1,3)} + \frac{1}{d(1,4)} + \frac{1}{d(1,5)} + \frac{1}{d(1,6)} + \frac{1}{d(1,7)} + \frac{1}{d(2,3)} + \frac{1}{d(2,4)} \\
&\quad + \frac{1}{d(2,5)} + \frac{1}{d(2,6)} + \frac{1}{d(2,7)} + \frac{1}{d(3,4)} + \frac{1}{d(3,5)} + \frac{1}{d(3,6)} + \frac{1}{d(3,7)} \\
&\quad + \frac{1}{d(4,5)} + \frac{1}{d(4,6)} + \frac{1}{d(4,7)} + \frac{1}{d(5,6)} + \frac{1}{d(5,7)} + \frac{1}{d(6,7)} \\
&= \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
&\quad + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
&= 13 \times \frac{1}{1} + 15 \times \frac{1}{2} \\
&= 13 + 7,5 \\
H(\Gamma_{\mathbb{Z}_8}) &= 13 + 7,5 = 20,5
\end{aligned}$$

**Lemma 2.10** [12] Given  $\Gamma_{\mathbb{Z}_n}$  a prime coprime graph of the group  $\mathbb{Z}_n$  and contains the vertex sets. Then  $\Gamma_{\mathbb{Z}_n}$  with  $n = p^k$  where  $p$  is a prime number and  $k \geq 2$  is an integer divided into two vertex sets.

$V_1$  contains the vertices in  $\mathbb{Z}_n$  that are multiples of  $p^{k-1}$ , that is:

$$V_1 = \{0, p^{k-1}, 2p^{k-1}, \dots, (p-1)p^{k-1}\}$$

$V_2$  contains all other vertices in  $\mathbb{Z}_n$  that are not multiples of  $p^{k-1}$ , that is:

$$V_2 = \{0, 1, 2, \dots, p^k - 1\} \setminus V_1$$

### 3 Result and Discussion

The structure of the prime coprime graph denoted by  $\Gamma_{\mathbb{Z}_n}$ , particularly when  $n = p^k$  where  $p$  is a prime number and  $k \geq 2$  is an integer, exhibits specific patterns of vertex connectivity. This structure is formally described in the following theorem:

**Theorem 3.1** Let  $\mathbb{Z}_n$  be a group with  $n = p^k$ , where  $p$  is a prime number and  $k \geq 2$  is an integer. Then, the graph of  $\Gamma_{\mathbb{Z}_n}$  has subgraphs with vertex sets  $V_1$  and  $V_2$ , where:

- $V_1$  contains  $p$  vertices that are adjacent to all vertices in  $V_1, V_2$ .
- $V_2$  contains  $p^k - p$  vertices that are adjacent only to all vertices in  $V_1$ .

PROOF: Based on Lemma 2.10, the set of vertices in  $\Gamma_{\mathbb{Z}_n}$  can be partitioned into two vertex sets is  $V_1$  and  $V_2$ :

- $V_1$  consists of all multiples of  $p^{k-1}$  in  $\mathbb{Z}_n$ , with the number of vertices  $|V_1| = p$ , since there are exactly  $p$  distinct multiples of  $p^{k-1}$  in  $V_1$ . For any two distinct vertices  $u_1, u_2 \in V_1$ ,  $|u_1| = |u_2| = p$ , so  $GCD(|u|, |v|) = p$ , which is a prime number. Then, all vertices in  $V_1$  and  $V_2$  are adjacent to vertices in  $V_1$ .
- $V_2$  consists of the remaining vertices, with  $|V_2| = p^k - |V_1| = p^k - p$ , for any  $v \in V_2$  has  $|v| = p^m$  for  $1 \leq m < k$ . Each vertex  $v \in V_2$  is adjacent to all vertices  $u \in V_1$  because  $GCD(p, p^m) = p$ , but for vertices  $v_1, v_2 \in V_2$ ,  $GCD(|v_1|, |v_2|) = p^{\min(m_1, m_2)}$ , which is not prime. Then no vertices in  $V_2$  are adjacent to each other.

Therefore,  $V_1$  forms a complete subgraph that is connected to all other vertices, while  $V_2$  is only connected to  $V_1$  and not to any vertices within  $V_2$

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**Theorem 3.2** Given  $\Gamma_{\mathbb{Z}_n}$  the prime coprime graph of  $\mathbb{Z}_n$ . If  $n = p^k$  with  $p$  prime number and  $k \geq 2$  is an integer, then the Schultz Index of  $\Gamma_{\mathbb{Z}_n}$  is:

$$S(\Gamma_{\mathbb{Z}_n}) = 3p^{2k+1} - 3p^{k+2} - 4p^{k+1} + p^3 + 2p^2 + p$$

PROOF: The proof is based on Theorem 3.1. We will divide the proof into three cases.

Case 3.2.1 for  $u, v \in V_1, u \neq v$

Since in Theorem 3.1 for  $\forall u, v \in V_1$  forms a complete subgraph of  $\Gamma_{\mathbb{Z}_n}$ , consequently  $d(u, v) = 1$ , for  $\forall u, v \in V_1$  with a total of  $p$  vertices, the number of possible pairs of vertices  $u, v$  is  $pC_2$ . Furthermore, because all vertices in  $V_1$  are adjacent to all vertices in  $V_1$  and  $V_2$  except with itself, then  $\deg(u) = p^k - 1$

Case 3.2.2 for  $u \in V_1, v \in V_2$

Since all vertices  $V_1$  are adjacent to all vertices in  $V_2$ , so  $d(u, v) = 1$ , for  $\forall u \in V_1, v \in V_2$ . And the possible pairs for the vertices  $u, v$  are obtained, namely:

$$\begin{aligned} \text{Number of pairs} &= |V_1| \times |V_2| \\ &= p \times (p^k - p) \\ &= (p^k - p)p \end{aligned}$$

Then, since all the nodes in  $V_2$  only adjacent by all nodes in  $V_1$  so  $\deg(v) = p$ , and  $\deg(u) = p^k - 1$ .

Case 3.2.3 for  $u, v \in V_2$

Since all vertices in  $V_2$  are not adjacent to each other and all of them adjacent to any vertex in  $V_1$ , it follows that  $d(u, v) = 2$ , for all  $\forall u, v \in V_2$ , with a total of  $p^k - p$ . Thus, the number of possible pairs of vertices  $u, v$  is  $(p^k - p)C_2$ . Furthermore, since all vertices in  $V_2$  are adjacent to all vertices in  $V_1$ , then  $\deg(v) = p$ .

Based on the Definition 2.4, the three cases above are summed, resulting in

$$\begin{aligned} S(\Gamma_{\mathbb{Z}_n}) &= \sum_{\{u,v\} \subset V_1} (\deg(u) + \deg(v)) d(u, v) + \sum_{u \in V_1, v \in V_2} (\deg(u) + \deg(v)) d(u, v) \\ &\quad + \sum_{\{u,v\} \subset V_2} (\deg(u) + \deg(v)) d(u, v) \\ &= (pC_2 [((p^k - 1) + (p^k - 1)).1]) + ((p^k - p)p [((p^k - 1) + (p)).1]) \\ &\quad + ((p^k - p)C_2 [((p) + (p)).2]) \\ &= \frac{p(p-1)}{2} [2((p^k - 1).1) + p^{k+1} - p^2[(p^k + p - 1).1] \\ &\quad + \frac{(p^k - p)(p^k - p - 1)}{2} [(2p).2] \\ &= p(p-1)(p^k - 1) + (p^k - p)p(p^k + p - 1) + 2p(p^k - p)(p^k - p - 1) \end{aligned}$$

$$\begin{aligned}
&= p^{k+2} - p^{k+1} - p^2 + p + p^{2k+1} - p^{k+1} - p^3 + p^2 + 2p^{2k+1} - 4p^{k+2} - 2p^{k+1} \\
&\quad + 2p^3 + 2p^2 \\
&= 3p^{2k+1} - 3p^{k+2} - 4p^{k+1} + p^3 + 2p^2 + p
\end{aligned}$$

Thus, the Schultz index is obtained:

$$S(\Gamma_{\mathbb{Z}_n}) = 3p^{2k+1} - 3p^{k+2} - 4p^{k+1} + p^3 + 2p^2 + p$$

■

**Theorem 3.3** Given  $\Gamma_{\mathbb{Z}_n}$  is the prime coprime graph of  $\mathbb{Z}_n$  with  $n = p^k$  where  $p$  is a prime number and  $k \geq 2$  is an integer. Then, the Forgotten Index of  $\Gamma_{\mathbb{Z}_n}$  is:

$$F(\Gamma_{\mathbb{Z}_n}) = p^{3k+1} - 3p^{2k+1} + 3p^{k+1} - p + p^{k+3} - p^4$$

PROOF: The proof is based on Theorem 3.1. We will divide the proof into two cases

Case 3.3.1 for  $v \in V_1$

Since the number of vertices in  $V_1$  is  $p$  and  $\deg(v) = p^k - 1$ ,

Case 3.3.2 for  $v \in V_2$

Since the number of vertices in  $V_2$  as much as  $p^k - p$  and  $\deg(v) = p$ ,

Thus, to find the Forgotten Index:

$$\begin{aligned}
F(\Gamma_{\mathbb{Z}_n}) &= \sum_{v \in V_1} \deg(v)^3 + \sum_{v \in V_2} \deg(v)^3 \\
&= ((p)(p^k - 1)^3) + ((p^k - p)(p)^3) \\
&= (p^{3k+1} - 3p^{2k+1} + 3p^{k+1} - p) + (p^{k+3} - p^4) \\
&= p^{3k+1} - 3p^{2k+1} + p^{k+3} + 3p^{k+1} - p^4 - 4 \\
&= p^{3k+1} - 3p^{2k+1} + p^{k+3} + 3p^{k+1} - p^4 - p
\end{aligned}$$

Thus, the Forgotten Index is obtained:

$$F(\Gamma_{\mathbb{Z}_n}) = p^{3k+1} - 3p^{2k+1} + p^{k+3} + 3p^{k+1} - p^4 - p$$

■

**Theorem 3.4** Given  $\Gamma_{\mathbb{Z}_n}$  is the prime coprime graph of  $\mathbb{Z}_n$ . If  $n = p^k$  with  $p$  a prime number and  $k \geq 2$  is an integer, then the Harary Index of  $\Gamma_{\mathbb{Z}_n}$  is:

$$H(\Gamma_{\mathbb{Z}_n}) = \frac{2p^{k+1} + p^{2k} - p^k - p^2 - p}{4}$$

PROOF: The proof is based on Theorem 3.1, we will divide the proof into two cases.

Case 3.4.1 for  $d(u, v) = 1$

Based on Theorem 3.2, in points 3.2.1 and 3.2.2, the number of possible pairs at distance 1

is  $(p^k - p)p + \frac{p(p-1)}{2}$ .

Case 3.4.2 for  $d(u, v) = 2$

The number of possible pairs at distance 2 is  $(p^k - p)C_2$ . Thus, to find the Harary Index obtained:

$$\begin{aligned}
 H(\Gamma_{\mathbb{Z}_n}) &= \sum_{\substack{\{u,v\} \in V_1 \\ d(u,v)=1}} \frac{1}{d(u,v)} + \sum_{\substack{\{u,v\} \in V_2 \\ d(u,v)=2}} \frac{1}{d(u,v)} \\
 &= \left( \left[ p(p^k - p) + \frac{p(p-1)}{2} \right] \cdot \frac{1}{1} \right) + \left( [(p^k - p)C_2] \cdot \frac{1}{2} \right) \\
 &= \left( p^{k+1} - p^2 + \frac{p^2 - p}{2} \right) + \left( \frac{(p^k - p)(p^k - p - 1)}{2} \cdot \frac{1}{2} \right) \\
 &= \frac{2p^{k+1} - 2p^2 + p^2 - p}{2} + \frac{p^{2k} - 2p^{k+1} - p^k + p^2 + p}{4} \\
 &= \frac{4p^{k+1} - 4p^2 + 2p^2 - 2p + p^{2k} - 2p^{k+1} - p^k + p^2 + p}{4} \\
 &= \frac{2p^{k+1} + p^{2k} - p^k - p^2 - p}{4}
 \end{aligned}$$

Thus, the Harary index is obtained:

$$H(\Gamma_{\mathbb{Z}_n}) = \frac{2p^{k+1} + p^{2k} - p^k - p^2 - p}{4}$$

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## 4 Conclusion

From the discussion results above, each of the indices analyzed—the Forgotten index, the Harary index, and the Schultz index—offers a distinct perspective on the structural properties of a graph. The Forgotten index emphasizes the strength of vertex connections based on their degrees, the Harary index focuses on the closeness or compactness of vertex connectivity, while the Schultz index integrates both degree and distance information to reflect relational dynamics within the graph. In the context of prime coprime graphs in the modulo  $\mathbb{Z}_n$  group for  $n = p^k$  where  $p$  is a prime number and  $k \geq 2$  is an integer, the indices are respectively  $F(\Gamma_{\mathbb{Z}_n}) = p^{3k+1} - 3p^{2k+1} + p^{k+3} + 3p^{k+1} - p^4 - p$ ,  $H(\Gamma_{\mathbb{Z}_n}) = \frac{2p^{k+1} + p^{2k} - p^k - p^2 - p}{4}$ , and  $S(\Gamma_{\mathbb{Z}_n}) = 3p^{2k+1} - 3p^{k+2} - 4p^{k+1} + p^3 + 2p^2 + p$ . These expressions demonstrate that the connectivity patterns in such graphs are heavily influenced by the divisibility properties of elements within the group. The algebraic structure of  $\mathbb{Z}_n$ , especially when  $n = p^k$  where  $p$  is prime number and  $k \geq 2$  is an integer, contributes significantly to shaping the graph's topology and thus directly impacts the values of these topological indices.

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