J. Ris. & Ap. Mat. Vol. 05 No. 02 (2021) pp. 118-124
Jurnal Riset dan Aplikasi Matematika
e-ISSN: 2581-0154
URL: journal.unesa.ac.id/index.php/jram

LESLIE MATRIX ANALYSIS IN DEMOGRAPHIC MODEL

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ABSTRACT

This study aims to show one of the benefits of algebra in modeling a problem. The population growth model that will be used is the Leslie Matrix. The Leslie matrix, also known as the Leslie model, was invented by P. H Leslie in 1945 to analyze population growth. It can be represented by the Perron root so that the Leslie model is simpler. The Perron root value affects the population, and If the Perron root value is greater than one, it means that the population will increase, while if the Perron root value is less than one, it means the population will decrease.

Key Word: leslie matrix, demographic, eigen value

1 Introduction

The science of demography consists of formal demography and population studies. In formal demography, techniques for calculating demographic measures were developed, such as adjusting data for estimates of population distribution by age, estimates of fertility, mortality and migration, and population projections. Formal demography uses mathematics and statistics as its analytical tools. Meanwhile, the analysis of the relationship between population and development aspects is studied in population studies [1].

One of the population growth models that demographers often use is the Lesli model. A population can be modeled with the Leslie Matrix by knowing the number of the female population. The composition of the number of women in a population is influenced by three factors, namely birth, death, and age.

In the Leslie Matrix, to find out the growth model of a population, several assumptions must be met, namely: only the female population is required, the maximum age that a population can reach, the population age group, the survival of each age group towards the next age stage is known and the birth rate for each age group is known [2].

Leslie Model has been used to predict woman population [2]–[4]. Modified Leslie-Gower predator-prey system with Crowley-Martin functional response and prey refuge [5]. The Leslie matrix to evolutionary demography used to the art of population projection models [6]. Leslie matrix by harvesting in the youngest age group (Harvesting the youngest class) and know the harvesting policy of each age group of sheep population has discussion on [7].

West Nusa Tenggara is one of the provinces that has the largest cattle population in Indonesia. Seeing the great potential in cattle farming, the NTB government since 2008 has launched the Bumi Sejuta Sapi program [8]. Based on the explanation above, in this research

2010 *Mathematics Subject Classification*: 05C50
Tanggal Masuk: 17-08-21; direvisi: 24-10-21; diterima: 29-10-21

the Leslie Model will be used in modeling and predicting the growth of cattle in West Nusa Tenggara Province in 2021.

2 Leslie Model

The female population is grouped into several age groups. Let U be the maximum age a female can give birth. Suppose the population is divided into n groups based on age. In that case, the interval distance for each group is U/n. Thus, the group 1 is those aged $\left[0, U/n\right)$, group 2 is that aged $\left[U/n, 2U/n\right)$ group n is those aged $\left[(n-1)U/n, U\right]$. Let the observation time be $t_i = 1, 2, 3, \cdots, k, \cdots$. In the Leslie model, the observation distance from t_{i-1} to t_i is the same as the group interval distance. So, $t_i = iU/n$ for $i = 1, 2, 3, \cdots$

Let a_i be the average number of girls born from each group i, and b_i is the ratio of the number of women who survive to enter the group $i+1$, with the number of women in the group i. It should be noted that at least one $a_i \geq 0, i = 1, 2, 3, \cdots$ and $0 < b_i < 1$. Note that at least one $a_i > 0$ because otherwise, it means that the birth process did not occur and $b_i > 0$ because otherwise, no female can survive into the group $i+1$.

We assume X_i^k to be the number of women in the group i at observations t_k. At the time of observation, t_k the number of girls in the first group is equal to the number of girls born in the first group from time t_{k-1} to t_k plus the number of girls born in the second group from time t_{k-1} to t_k plus the number of girls born in the group n from time t_{k-1} to t_k. So that,

$$X_1^k = a_1 X_1^{k-1} + a_2 X_2^{k-1} + \cdots + a_n X_n^{k-1} \qquad (1)$$

Because the interval distance of each group is equal to the distance of two consecutive observations, all females who were in the group $i+1$ at the time of observation t_{k+1} were in the group i at the time of observation t_k. Therefore, the number of females in the group $i+1$ at the time of observation t_k is the same as the number of females who are still alive in the group i at the time t_{k-1} to t_k. So that,

$$\begin{aligned}
X_1^k &= a_1 X_1^{k-1} + a_2 X_2^{k-1} + \cdots + a_n X_n^{k-1} \\
X_2^k &= b_1 X_1^{k-1} \\
X_3^k &= b_2 X_2^{k-1} \\
&\vdots \\
X_i^k &= b_{i-1} X_{i-1}^{k-1}, \ i = 1, 2, 3, \cdots, n-1 \qquad (2)
\end{aligned}$$

Equations (1) and (2) can be written in matrix form

$$X^k = L X^{k-1} \quad k = 1, 2, 3, \cdots$$

$$\begin{bmatrix} X_1^k \\ X_2^k \\ X_3^k \\ \vdots \\ X_n^k \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ b_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_n & 0 \end{bmatrix} \begin{bmatrix} X_1^{k-1} \\ X_2^{k-1} \\ X_3^{k-1} \\ \vdots \\ X_n^{k-1} \end{bmatrix}$$

The vector X^k is the age distribution vector at time t_k , and the vector X^{k-1} is the age distribution vector at the time t_{k-1} , while the matrix L is called the Leslie matrix [9].

Leslie's matrix is a non-negative matrix that has positive eigenvalues. Then the largest positive eigenvalue is called the Perron root. The following theorem explains the relationship between the Leslie matrix and the Perron root.

Theorem If λ_p is the Perron root of the Leslie matrix, L then

$$X^k = L\ X^{k-1}$$

can be represented as $X^k = \lambda_p\ X^{k-1}$.

Proof The eigenvalues of the Leslie Matrix can be found using the characteristic equation $\lambda = \det(L - \lambda I)$. As Leslie matrix L is diagonalizable, then $L = VDV^{-1}$ where D is a diagonal matrix whose elements are the eigenvalues of L and V is an invertible matrix constructed by the corresponding eigenvectors. So that,

$$L = V\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} V^{-1} \tag{3}$$

where

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \tag{4}$$

By multiplying k factors, we get the equation

$$L^k = \left(VDV^{-1}\right)^k = \underbrace{\left(VDV^{-1}\right)\left(VDV^{-1}\right)\cdots\left(VDV^{-1}\right)}_{k\ factor} = VD^kV^{-1}$$

So that,

$$L = V\begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} V^{-1}$$

For any age vector, x^0 we can find the age vector x^k after k years by finding $L^k x^0$

$$x^k = L^k x^0 = V\begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} V^{-1}\, x^0 \tag{5}$$

We assume,

$$V^{-1}x^0 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then equation (5) can be written as

$$x^k = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ 0 \\ c_n \end{bmatrix}$$

So that

$$x^k = 1\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} c_1\lambda_1^k \\ c_2\lambda_2^k \\ \vdots \\ c_n\lambda_n^k \end{bmatrix} \qquad (6)$$

Thus,

$$x^k = c_1\lambda_1^k v_1 + c_2\lambda_2^k v_2 + \cdots + c_n\lambda_n^k v_n \qquad (7)$$

Suppose λ_1 is the largest eigenvalue of any existing eigenvalues (λ_1 is strictly dominant eigenvalue). Then if equation (7) is divided by λ_1^k, we get

$$\frac{x^k}{\lambda_1^k} = \frac{\lambda_1^k}{\lambda_1^k} c_1 v_1 + \frac{\lambda_2^k}{\lambda_1^k} c_2 v_2 + \cdots + \frac{\lambda_n^k}{\lambda_1^k} c_n v_n \qquad (8)$$

If $\left|\lambda_1^k\right| > \left|\lambda_i^k\right|$ $i = 2,3,\cdots,n$ then $\left|\frac{\lambda_i^k}{\lambda_1^k}\right| < 1$

So that,

$\left(\frac{\lambda_i}{\lambda_1}\right)^k \longrightarrow 0$ if $k \longrightarrow \infty$ for $i = 2,3,\cdots,n$

Based on this, if the limits are taken for both sides, then from equation (8)

$$\lim_{k\to\infty}\left(\frac{x^k}{\lambda_1^k}\right) = \lim_{k\to\infty}\left(\frac{\lambda_1^k}{\lambda_1^k} c_1 v_1 + \frac{\lambda_2^k}{\lambda_1^k} c_2 v_2 + \cdots + \frac{\lambda_n^k}{\lambda_1^k} c_n v_n\right)$$

$$\Leftrightarrow \lim_{k\to\infty}\left(\frac{x^k}{\lambda_1^k}\right) = c_1 v_1 + \lim_{k\to\infty}\left(\frac{\lambda_2^k}{\lambda_1^k} c_2 v_2 + \frac{\lambda_3^k}{\lambda_1^k} c_3 v_3 + \cdots + \frac{\lambda_n^k}{\lambda_1^k} c_n v_n\right)$$

$$\Leftrightarrow \lim_{k\to\infty}\left(\frac{x^k}{\lambda_1^k}\right) = c_1 v_1$$

Since c_1v_1 is a constant, it implies that

$$\lim_{k\to\infty}\left(\frac{x^k}{\lambda_1^k}\right)=\lim_{k\to\infty}c_1v_1$$

$$\Leftrightarrow \lim_{k\to\infty}\left(\frac{x^k}{\lambda_1^k}-c_1v_1\right)=0$$

It means

$$\left|\frac{x^k}{\lambda_1^k}-c_1v_1\right|<\varepsilon$$

in other words

$$0<\frac{x^k}{\lambda_1^k}-c_1v_1<\varepsilon$$

So,

$$\frac{x^k}{\lambda_1^k}-c_1v_1=0$$

For a large value of k the approximate value for x^k is up to

$$x^k\approx\lambda_1^k c_1v_1 \qquad (9)$$

For $k-1$ applies

$$x^{k-1}\approx\lambda_1^{k-1}c_1v_1$$

So that,

$$v_1\approx\frac{x^{k-1}}{\lambda_1^{k-1}c_1}$$

From here, equation (9) becomes ([2], [7])

$$x^k\approx\lambda_1^k c_1\frac{x^{k-1}}{\lambda_1^{k-1}c_1}$$

$$x^k\approx\lambda_1 x^{k-1}$$

3 Result and Discussion

Based on the results of the analysis of cattle population data from 2014 to 2020, data on the composition of cattle were obtained as follows [10]:

Tabel : COW COMPOSITION IN NTB

Year	Population	AGE GROUP (FEMALE)		
		Kids	Young	Adult
2014	1013793	132566	126802	314415
2015	1055013	137956	131958	327199
2016	1092719	142887	136674	338893
2017	1149539	150316	143781	356515
2018	1183570	154766	148037	367069
2019	1234640	161444	154425	382908
2020	1285746	168127	160817	398758

From the data on the composition of cows, the child's survival rate is 1.00, and the young age's survival is 2.58. Furthermore, the obtained survival rate is 0.44. Leslie matrix is formed as follows:

$$L = \begin{bmatrix} 0 & 0 & 0,44 \\ 1 & 0 & 0 \\ 0 & 2,58 & 0 \end{bmatrix}$$

The values of $a_1 = 0$ and $a_2 = 0$ are due to the absence of births in the calves and young cattle groups. The $a_3 = 0,44$ value is the birth rate for the adult cattle group. The $b_1 = 1$ value is the ratio of the number of female cows in the young age group in the year t to the number of female cows in the child age group $t-1$. Meanwhile, $b_2 = 2,58$ is the ratio of the number of female cows in the adult age group in the year t to the number of female cows in the young age group in the year $t-1$. Perron root of the matrix is $\lambda_p = 1,04$ so that the cattle growth model in West Nusa Tenggara is:

$$X^k = L\,X^{k-1} \quad k = 1, 2, 3, \cdots$$

$$X^k = \begin{bmatrix} 0 & 0 & 0,44 \\ 1 & 0 & 0 \\ 0 & 2,58 & 0 \end{bmatrix} X^{k-1}$$

Based on Theorem, Leslie Matrix L above can be represented by a constant, namely the Perron root of the matrix so that the model becomes

$$X^k = \lambda_p \; X^{k-1}$$

$$X^k = 1,043 \; X^{k-1}$$

The Leslie model above can predict the cattle population in the coming year.

4 Conclusion

It can be concluded that the Leslie L matrix in the Leslie model $X^{k} = L\ X^{k-1}$ can be represented by the Perron root so that the Leslie model becomes $X^{k} = \lambda_p X^{k-1}$. Perron root value affects the number of populations if $\lambda_p > 1$ it means the population will increase while if $\lambda_p < 1$ means the population will decrease. Leslie model for the cattle population is $X^{k} = 1{,}043\ X^{k-1}$.

5 Acknowledgement

The author would like to thank the University of Bangka Belitung for the opportunity to research and training in writing articles.

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