J. Ris. & Ap. Mat. Vol. 4 No. 2 (2020) pp. 103-107
Jurnal Riset dan Aplikasi Matematika
e-ISSN: 2581-0154
URL: journal.unesa.ac.id/index.php/jram

SOME NOTES ON A GENERALIZED VERSION OF PYTHAGOREAN TRIPLES

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ABSTRACT

A Pythagorean triple is a set of three positive integers a, b and c that satisfy the Diophantine equation $a^2 + b^2 = c^2$. The triple is said to be primitive if $\gcd(a, b, c) = 1$ and each pair of integers $a, b,$ and c are relatively prime, otherwise known as non-primitive. In this paper, the generalized version of the formula that generates primitive and non-primitive Pythagorean triples that depends on two positive integers k and n, that is, $P_T = (a(k,n), b(k,n), c(k,n))$ were constructed. Further, we determined the values of k and n that generates primitive Pythagorean triples and give some important results.

Keywords: Pythagorean triple, Diophantine equation, Primitive

1 Introduction

In elementary number theory, the idea of Pythagorean triples remains intriguing and receiving much attention to research in discrete mathematics [1] [2] [3]. A Pythagorean triple (a, b, c) is a triple of positive integers such that $a^2 + b^2 = c^2$ [4] [5]. Some Pythagorean triples are scalar multiples of other triples such as (6, 8, 10) is twice (3, 4, 5). A primitive Pythagorean triple is one in which any two of the three numbers are relatively prime. A Pythagorean prime is a prime number of the form 4n + 1. Pythagorean primes are exactly the odd prime numbers that are the sum of two squares [6]. The method of generating Pythagorean triples is known for about 2000 years [7]. Though the classical formulas produce all primitive triples they do not generate all possible triples, especially non-primitive triples. Hence, in this paper we construct the generalized version of the formula that generates primitive and non-primitive Pythagorean triples that depends on two positive integers k and n, that is, $P_T = (a(k,n), b(k,n), c(k,n))$. Further, we determine the values of k and n that generates primitive Pythagorean triples and discuss some important results. The purpose of the paper is to understand the new mathematical concepts of Pythagorean triples.

2 Methodology

The methodology of this study is based on the recent paper by Casinillo [8] that deals with an exploratory in nature. Firstly, the formula that generates Pythagorean triples depending on two positive integers was presented as the characterization theorem. Secondly, the proof of the characterization theorem of generalized Pythagorean triples was presented. Lastly, this paper developed new results for primitive and non-primitive Pythagorean triples with detailed proofs which is a consequence of the characterization theorem. Figure 1 presents the schematic diagram of the study.

2010 *Mathematics Subject Classification*: 11A07, 11A41, 11D45
submitted: 08-09-20; revised: 20-10-20; accepted: 21-10-20

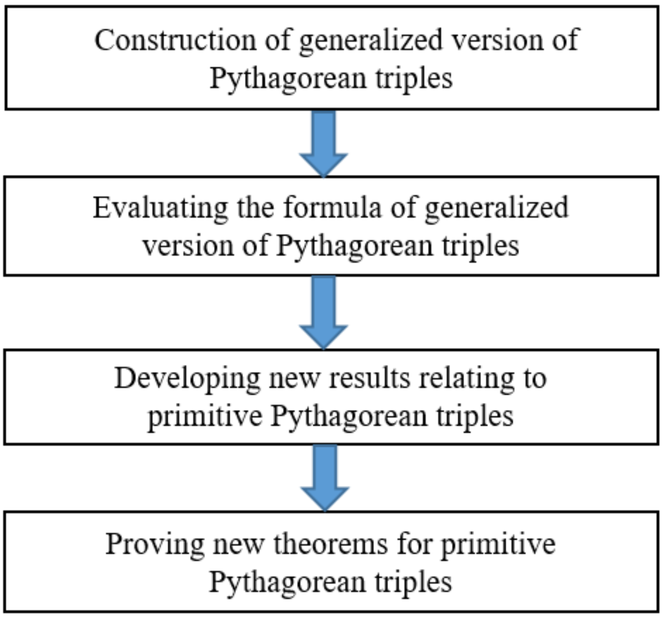

Figure 1. Schematic Diagram of the Study

3 Results

The following Theorem is a new generalized version that generates primitive and non-primitive Pythagorean triples for all pairs of positive integers.

Theorem 1. For any positive integers k and n, $(2n^2 + 2kn,\ 2kn + k^2,\ 2n^2 + 2kn + k^2)$ is a Pythagorean triple. Conversely, for any Pythagorean triple (a, b, c), there are positive integers k and n such that $a = 2n^2 + 2kn,\ b = 2kn + k^2$ and $c = 2n^2 + 2kn + k^2$.

PROOF. $(\Rightarrow)$ Suppose that $a = 2n^2 + 2kn,\ b = 2kn + k^2$ and $c = 2n^2 + 2kn + k^2$ for all positive integers n and k. Then,

$$a^2 + b^2 = (2n^2 + 2kn)^2 + (2kn + k^2)^2 \tag{2.1}$$
$$= 4n^4 + 8kn^3 + 4k^2n^2 + 4k^2n^2 + 4k^3n + k^4 \tag{2.2}$$
$$= k^4 + 4k^3n + 8k^2n^2 + 8kn^3 + 4n^4 \tag{2.3}$$
$$= (2n^2 + 2kn + k^2)^2 \tag{2.4}$$
$$= c^2 \tag{2.5}$$

$(\Leftarrow)$ For the converse, we suppose that a, b and c are positive integers and satisfies the Diophantine equation $a^2 + b^2 = c^2$. Then, we have

$$1 = \frac{(c+b)(c-b)}{a^2} \tag{2.6}$$

And it follows that,

$$1 = \left(\frac{c}{a} + \frac{b}{a}\right)\left(\frac{c}{a} - \frac{b}{a}\right) \tag{2.7}$$

Now, we let n and k be positive integers such that

$$\frac{c}{a} + \frac{b}{a} = \frac{n+k}{n} \tag{2.8}$$

and

$$\frac{c}{a} - \frac{b}{a} = \frac{n}{n+k} \tag{2.9}$$

Solving for $\frac{b}{a}$ in (2.8), we obtained,

$$\frac{b}{a} = \frac{n+k}{n} - \frac{c}{a} \tag{2.10}$$

Substituting (2.10) to (2.9), we have,

$$\frac{c}{a} - \left(\frac{n+k}{n} - \frac{c}{a}\right) = \frac{n}{n+k} \tag{2.11}$$

Simplifying the equation we end up with,

$$\frac{c}{a} = \frac{2n^2 + 2kn + k^2}{2n^2 + 2kn} \tag{2.12}$$

Substituting (2.12) to (2.10) and simplifying the equation, we also end up with,

$$\frac{b}{a} = \frac{2kn + k^2}{2n^2 + 2kn} \tag{2.13}$$

Thus, this implies that $a = 2n^2 + 2kn,\ b = 2kn + k^2$ and $c = 2n^2 + 2kn + k^2$. This completes the proof. ■

The generalized Pythagorean triples $a(k,n), b(k,n)\ and\ c(k,n)$ in Theorem 1 was evaluated to its mathematical property. Hence, the next Corollary is immediate from Theorem 1, showing the inequality relationship of the said triples in view of Trichotomy Property of Order of positive integers k and n.

Corollary 2. Let k and n be positive integers. If $a(k,n), b(k,n)\ and\ c(k,n)$ are generalized Pythagorean triples, then the following holds true:

$c(k,n) > b(k,n) > a(k,n) \geq 6$ where $n < k$; and

$c(k,n) > a(k,n) > b(k,n) \geq 3$ where $n \geq k$.

The following Corollary is a quick consequence of Theorem 1, Wilson's Theorem [4] [5] and the definition of Pythagorean prime above.

Corollary 3. Let $\rho(c) = cos^2\left[\frac{\pi(c-1)!+1}{c}\right]$ and $a^2 + b^2 = c^2$ for some positive integer a, b and c. If $\rho(c) = 1,$ then the following hold: (i) $P_T = (a,b,c)$ is a primitive Pythagorean triple; and (ii) c is a Pythagorean prime.

Our argument prior to the statement of Theorem 1 is that primitive Pythagorean triple can be generated for some positive integers n and k. In view of Theorem 1, the following results are immediate.

Theorem 4. If $n = 2^\alpha$ and $k = 2^\beta$, where α and β are positive integers, then $gcd\big(a(k,n), b(k,n), c(k,n)\big) = 2^{min\,(\alpha+\beta+1,\ 2\alpha+1,\ 2\beta)}$.

PROOF. Let $n = 2^\alpha$ and $k = 2^\beta$ for positive integers α and β. Then, by Theorem 1, we obtained $a = 2(2^\alpha)^2 + 2(2^\beta 2^\alpha),\ b = 2(2^\beta 2^\alpha) + \left(2^\beta\right)^2$ and $c = 2(2^\alpha)^2 + 2(2^\beta 2^\alpha) + \left(2^\beta\right)^2$. It follows that $a = 2^{\alpha+\beta+1} + 2^{2\beta},\ b = 2^{2\alpha+1} + 2^{\alpha+\beta+1}$ and $c = 2^{2\alpha+1} + 2^{\alpha+\beta+1} + 2^{2\beta}$. Thus, this implies that $\gcd\big(a(k,n), b(k,n), c(k,n)\big) = 2^{\min(\alpha+\beta+1,\ 2\alpha+1,\ 2\beta)}.T$ his completes the proof. ■

The next result is a direct consequence of Theorem 4.

Corollary 5. If $n = 2^{\alpha}$ and $k = 2^{\beta}$, where α and β are positive integers and $d = 2^{min\,(\alpha+\beta+1,\ 2\alpha+1,\ 2\beta)}$. Then, $P_T = \frac{1}{d}(a(k,n), b(k,n), c(k,n))$ is a primitive Pythagorean triple.

Our next theorems determine a primitive Pythagorean triple using the parity of positive integers n and k.

Theorem 6. Let $n \equiv 1(mod\ 2)$ and $k \equiv 1(mod\ 2)$ and $n \neq k$. Then, $P_T = (a(k,n), b(k,n), c(k,n))$ is a primitive Pythagorean triple.

PROOF. Let $n = 2x + 1$ and $k = 2y + 1$ where x and y are positive integers. By Theorem 1, it follows directly that, $a = 2(2x+1)^2 + 2(2y+1)(2x+1)$, $b = 2(2y+1)(2x+1) + (2y+1)^2$ and $c = 2(2x+1)^2 + 2(2y+1)(2x+1) + (2y+1)^2$.
Note that $n \neq k$, then by inspection, a, b and c has no common factor greater than 1. Thus, a, b and c are relatively prime. This completes the proof. ■

Theorem 7. Let $n = 1$ and $k \equiv 0(mod\ 2)$. Then, $gcd(a(k,n), b(k,n), c(k,n)) = 2$.

PROOF. Let $n = 1$ and $k = 2a$, for all positive integer a. Then, by Theorem 1, we obtained, $a = 2 + 2(2a), b = 2(2a) + (2a)^2$ and $c = 2 + 2(2a) + (2a)^2$. Then, rewriting the three equations, it follows that, $a = 2(1 + 2a), b = 2(4a)$ and $c = 2(1 + 2a + 2a^2)$. Thus, the greatest common factor is 2 and this completes the proof. ■

Corollary 8. If $n = 1$ and $k \equiv 0(mod\ 2)$, then $P_T = \frac{1}{2}(a(k,n), b(k,n), c(k,n))$ is a primitive Pythagorean triple.

Theorem 9. Let μ be a positive integer. If $n = k = \mu$, then,

$$gcd(a(k,n), b(k,n), c(k,n)) = \mu^2.$$

PROOF. Let μ be a positive integer and $n = k = \mu$. By Theorem 1, it follows that $a = 4\mu^2$, $b = 3\mu^2$ and $c = 5\mu^2$. And it is easy to check that the greatest common divisor is μ^2. This completes the proof. ■

Next, the following Corollary and Remark shows a primitive Pythagorean triple which is a direct consequence of Theorem 9.

Corollary 10. If $n = k = \mu$, then $P_T = \frac{1}{\mu^2}(a(k,n), b(k,n), c(k,n))$ is a primitive Pythagorean triple.

Remark 11. If $n = k = \mu$, then, $c = a + \mu^2 = b + 2\mu^2$.

The next result will generate Pythagorean triples in which the longer leg of the triangle and hypotenuse differ by one.

Theorem 12. If $k = 1$ and n be any positive integer, then, $P_T = (a(k,n), b(k,n), c(k,n))$ is a primitive Pythagorean triple with $c - 1 = a > b \geq 3$.

PROOF. Suppose that $k = 1$ and n is any positive integer. Then by Theorem 1, it implies that $a = 2n^2 + 2n$, $b = 2n + 1$ and $c = 2n^2 + 2n + 1$ and $\gcd(a,b,c) = 1$.

This follows that $P_T = (a(n,k), b(n,k), c(n,k))$ is a primitive Pythagorean triple. Also, for any positive integer n we have, $2n^2 + 2n + 1 > 2n^2 + 2n > 2n + 1 \geq 3$. And this completes the proof. ■

The following Corollary is a direct consequence of Theorem 12 that determines the parity of the Pythagorean triples. Parity is interesting topic in number theory which refers to a relationship of oddness or evenness between integers. Determining the parity of the new discovered numbers even exists in the paper of Casinillo [9].

Corollary 13. If $k = 1$ and n be any positive integer, then, $a(k,n) \equiv 0(mod\ 2), b(k,n) \equiv 1(mod\ 2)$ and $c(k,n) \equiv 1(mod\ 2)$.

4 Conclusions

This paper developed a new formula that presents a generalized version of Pythagorean triples which leads to a characterization theorem. The formula constructed generates non-primitive and primitive Pythagorean triples. Some new results on primitive Pythagorean triples were developed as a consequence of the said characterization theorem and these results were provided with a mathematical proof. In future research, one may consider a concept of congruent numbers that relates the new formula for generalized version of Pythagorean triples.

5 Acknowledgements

We would like to acknowledge the referees for the rigorous review and comments for the improvement of this paper.

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