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SYMMETRICAL MATRICES CHARACTERISTIC OF INTEGERS WITH INTEGER EIGEN VALUE

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ABSTRACT

Estes (1992) stated that the set of eigen values from symmetrical matrices of Z is a set of totally real algebraic integers. Estes was not able to ensure that eigen values of a symmetrical matrices are integers. Mckee and Smyth (2007) observed more about the eigen value of symmetrical integer matrices. James and Chris proved that symmetrical integer matrices have eigen value with interval ranging in [-2,2]. Contrary to that, Martin and Wong (2009), stated that almost all integer matrices have no integer eigen value. Previous studies that could not show the characteristic of the eigen value made Cao and Koyunco studied and tried to determine the characteristic of symmetrical integer matrices for rank 2 and rank 3. The result shows that they have integer eigen value. In accordance to Cao and Kuyonco study, this article elaborates the characteristic of a symmetrical integer matrices for rank 4, and 5 to show the characteristic of a symmetrical integer matrices with integer eigen value for rank 1, 2, 3 and for rank 4 and 5.

Keywords: symmetrical matrices, rank, polynomial characteristic, and eigen value.

1. Introduction

There have been so many studies about matrix that have been conducted by mathematicians because the theory of matrix as one of linier algebra branch that has an essential role in the study of mathematics. Particularly, those studies are about the nature and characteristics of a symmetric matrix $A \in M_n(Z)$. A square matrix $A = [a_{ij}]$ is called being symmetric if for all i and j, $a_{ij} = a_{ji}$. By considering this formula, it is clearly called a symmetric matrix if it is a square matrix and its each element $a_{ij} = a_{ji}$. In other words, it fulfills the condition $A^T = A$.

The discussion about matrix cannot be separated from dimensions, row spaces, column spaces, matrix rank, nullity and characteristic polynomials. The characteristic polynomial will lead to find the eigenvalues of a matrix that fulfils the equation: $\det(\lambda I - A) = 0$.

One of the most frequently asked in linier algebra is: if $A \in M_{n\times n}A$ is square matrix so all the eigenvalues are real. Due to its interesting discussion, many mathematicians' studies on how the characteristics of a symmetric matrix over Z getting integers in its eigenvalues.

For several years, the discussion about eigenvalues of a symmetric values keeps being studied by many researchers. In 1992, D.R.Estes in his paper entitled "Eigenvalues of Symmetric Integer Matrices" has proven that the set of eigenvalues $E(Z)$ of a symmetric matrix over Z is showed as totally real algebraic integers. However, Estes has not been able to ensure that the eigenvalues from symmetric matrix $A \in M_n(Z)$ is an integer.

Then, in 2007, James Mckee dan Chris Smyth studied deeper about the eigenvalues of integer symmetric matrix. In their paper, "Integer Symmetric Matrices Having All Their

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Eigenvalues in the interval $[-2,2]$". James and Chris proved that integer symmetric matrices that have eigenvalues in the interval $[-2,2]$.

On the other hand, in 2009, G. Martin and E.B Wong, Amer in their paper entitled "Almost All Integer Matrices Have No Integer Eigenvalues" precisely stated that almost all integer matrices have no integer eigenvalues.

Due to its unsureness of the integer eigenvalues from previous related studies, Lei Cao and Selcuk Kuyonco in 2016 finally studied and sought to the characteristics of integer symmetric matrix in rank two dan three that have eigen integer, that realized on their paper: "Symmetric Integer Matrices Having Integer Eigenvalues".

2. Literature Review

The research problem of this research is the characteristics of integer symmetric matrices by knowing the matrix rank. Then it will be shown that the matrix has integer eigenvalues. The significances of this researchers are as follows.

1. Investigating how the characteristics of integer symmetric matrices in rank 1,2, and 3 that have integer eigenvalues.
2. Investigating whether the characteristics of an integer symmetric matrix in rank 4 and 5 still fulfills the core theory in point 1.

This study is limited to the square matrix with size $n \times n$ in rank 1, 2, and 3. Then, the matrix used in next discussion is the square matrix with size $n \times n$ in rank 4 and 5.

Theorem 1. If $A \in M_{n\times n}$ is symmetric matrix so all the eigenvalues are real. (Zdeněk Dvořák 2016)

Proof.

Given A symmetric matrix, means $A = A^T$. If $\lambda \in C$ is a eigenvalues from symmetric matrix A. So, $Av = \lambda v$, $v \neq 0$, and

$$\overline{Av} = \overline{\lambda v} \Longrightarrow A\bar{v} = \bar{\lambda}\bar{v}$$

Because of A is symmetric:

$$\bar{v}^T Av = \bar{v}^T(Av) = \bar{v}^T(\lambda v) = \lambda(\bar{v}\,.\,v)$$
$$\bar{v}^T Av = (A\bar{v})^T v = \left(\bar{\lambda}\bar{v}\right)^T v = \bar{\lambda}(\bar{v}\,.\,v)$$

As $v \neq 0$ means $\bar{v}\,.\,v \neq 0$.

Hence, it must be $\lambda = \bar{\lambda}$. This means $\lambda \in \mathcal{R}$.

Theorem 2. If $A \in M_n(Z)$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Polynomial characteristics of A :

$$P_A(\lambda) = \det(\lambda I - A) \qquad = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0$$

so :

$$c_i = A_{[i-n]},\ i = 0,1, \dots, n-1$$

(Denton et al. 2022).

Proof.

If $A \in M_n(Z)$ with eigenvalues $\lambda_1, \lambda_{2, \dots,} \lambda_n$. The polynomial characteristics of A :

$$P_A(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0.$$

The goal is to find the coefficients of every polynomial characteristic.

Take $A \in M_2(Z)$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

So , the polynomial characteristics is:

$$P_2(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

With

$$a_{11} + a_{22} = trace(A)$$

$$(a_{11}a_{22} - a_{12}a_{21}) = A_{[2]}$$

Hence, generally:

$$P_2(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - trace(A)\lambda + A_{[2]}$$

Take $A \in M_3(Z), A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

So, the polynomial characteristics is:

$$P_3(\lambda) = \lambda^3 - trace(A)\lambda + A_{[2]}\lambda - A_{[3]}$$

Hence, generally, $P_A(\lambda) = \lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + \cdots + (-1)^n c_n$
where $a_i = \sum_i a_{ii}$ is $trace(A)$ and $c_n = \det(A)$, proves the sum of $n - minor$.

Lemma 1. Let $A \in M_n(Z)$ with rank 1. Then A has integer eigenvalue. (Akiyama et al. 2006)
Proof.
Let $A \in M_n(Z)$ with rank 1. Then the characteristic polynomial of A is:

$$\lambda^n - trace(A)\lambda^{n-1} = 0$$
$$\lambda^{n-1}\big(\lambda - trace(A)\big) = 0$$
$$\lambda - trace(A) = 0$$
$$\lambda = trace(A)$$

Because $A \in M_n(Z)$, so $trace(A) = k \in Z$. Hence, eigenvalue λ is integers.
The main theorem to be proved is.

3. Result

Theorem 3. Let $A \in M_n(Z)$ with rank 2, so A has integer eigenvalue if and only if there exist two integers $m, n \in Z$ such that $trace(A) = m + n$ dan $A_{[2]} = m * n$, where $A_{[2]}$ is the sum of determinants of all 2nd order principal minors of A.
Proof.
Since A has rank 2, the characteristic polynomial of A has the form:

$$\lambda^n - trace(A)\lambda^{n-1} + A_{[2]}\lambda^{n-2}$$

Based on the formula to solve the quadratic equation, it is obtained:

$$\lambda_{1,2} = \frac{trace(A) \pm \sqrt{trace(A)^2 - 4A_{[2]}}}{2}$$

($\Rightarrow$) If there exist two integers $m, n \in Z$ such that $trace(A) = m + n$ dan $A_{[2]} = m * n$. This means is $\lambda_1 = m$, and $\lambda_2 = n$.
($\Leftarrow$) If A has integer eigenvalue, so there exists $k \in Z$ such that:

$$trace(A)^2 - 4A_{[2]} = k^2 \qquad (*)$$

Let $m = \frac{trace(A)+k}{2}, n = \frac{trace(A)-k}{2}$
Look at $(*)$, $trace(A)^2 - k^2 = 4A_{[2]}$ is even, so $trace(A)$ dan k are both even or odd. That means, m and n are integers.

Theorem 4. Let $A \in M_n(Z)$ with rank 3. If one of the following cases hold, then A has integer eigenvalues.
(i) One of the eigenvalues of A is 1 or -1 and there exists a positive integer $k \in Z$ such that:

$$\left[A_{[3]} - A_{[2]}\right]^2 - 4A_{[3]} = k^2$$

(ii) All nonzero eigenvalues of A are the same and

$$A_{[2]} = \frac{trace(A)^2}{3}, \qquad A_{[3]} = \frac{trace(A)^3}{27}$$

(iii) One of the nonzero eigenvalues of A has multiplicity two and there exists a positive integer $k \in Z^{+}$such that: $trace(A)^2 - 3A_{[2]} = k^2$

(iv) $Trace(A) = 0$ and there exists a positive integer $k \in Z^{+}$ and $m, n \in Z$ such that:

$$k = \sqrt{\frac{\left(A_{[3]}\right)^2}{4} + \frac{\left(A_{[2]}\right)^3}{27}}$$

$$m^3 = \frac{A_{[3]}}{2} + k, \;\; n^3 = \frac{A_{[3]}}{2} - k$$

In fact, one of eigenvalues $m + n$.

Proof.

Let $A \in M_n(Z)$ with rank 3, so the characteristics polynomial of A can be written as

$$P_A(\lambda) = \lambda^n - trace(A)\lambda^{n-1} + A_{[2]}\lambda^{n-2} - A_{[3]}\lambda^{n-3}$$
$$= \lambda^{n-3}\left(\lambda^3 - trace(A)\lambda^2 + A_{[2]}\lambda - A_{[3]}\right) \; (*)$$

(i) Suppose one of the eigenvalues of A is 1, then:

$$(1)^3 - trace(A)(1)^2 + A_{[2]} - A_{[3]} = 0$$
$$1 - trace(A) = A_{[3]} - A_{[2]}$$

Look at $(*)$, it can be factored into:

$$P_A(\lambda) = \lambda^{n-3}(\lambda - 1)\left[\lambda^2 + \left(1 - trace(A)\right)\lambda + A_{[3]}\right]$$

By theorem 1, the quadratic factor has integer roots if and only if there exists a positive integer $k \in Z^{+}$ such that:

$$\left(1 - trace(A)\right)^2 - 4A_{[3]} = k^2$$

In other hands:

$$1 - trace(A) = A_{[3]} - A_{[2]}$$

so:

$$\left(A_{[3]} - A_{[2]}\right)^2 - 4A_{[3]} = k^2$$

Then the other eigenvalues are:

$$\lambda_2 = \frac{-(1 - trace(A)) + k}{2}$$
$$\lambda_3 = \frac{-(1 - trace(A)) - k}{2}$$

Because $\left(1 - trace(A)\right)^2 - k^2 = 4A_{[3]}$ is even, hence $1 - trace(A)$ and k are both even or odd. Hence, λ_2, λ_3 are integers.

(ii) The non-zero eigenvalues of A are equal and

$$A_{[2]} = \frac{trace(A)^2}{3}, \qquad A_{[3]} = \frac{trace(A)^3}{27}$$

Let $\tilde{\lambda}$ the only non-zero eigenvalue of A, so:

$$P_A(\lambda) = \lambda^{n-3}\left(\lambda - \bar{\lambda}\right)^3$$
$$\left(\lambda^3 - trace(A)\lambda^2 + A_{[2]}\lambda - A_{[3]}\right) = \left(\lambda - \bar{\lambda}\right)^3$$

Hence it is obtained:

$$trace(A) = 3\bar{\lambda}$$
$$A_{[2]} = 3\bar{\lambda}^2$$
$$A_{[3]} = \bar{\lambda}^3$$

Because $trace(A) = 3\tilde{\lambda}$, maka $\tilde{\lambda} = \frac{trace(A)}{3}$, such that

$$A_{[2]} = \frac{trace(A)^2}{3}, \qquad A_{[3]} = \frac{trace(A)^3}{27}$$

Due to rank 1 case, $trace(A) = k, k \in Z$, so

$$A_{[2]} = \frac{\left(trace(A)\right)^2}{3} = \frac{k^2}{3}$$

and

$$A_{[3]} = \frac{\left(trace(A)\right)^3}{27} = \frac{k^3}{27}$$

(iii) One of the nonzero eigenvalues of A has multiplicity two and there exists a positive integer $k \in Z^+$such that: $trace(A)^2 - 3A_{[2]} = k^2$.

For example, A has non-zero eigenvalues, namely λ_1 and λ_2 with a multiplicity two. This can be written as:

$$P_A(\lambda) = \lambda^{n-3}(\lambda - \lambda_1)^2(\lambda - \lambda_2).$$

so:

$$\lambda^3 - trace(A)\lambda^2 + A_{[2]}\lambda - A_{[3]} = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$$

Hence, it is obtained:

$$trace(A) = 2\lambda_1 + \lambda_2$$
$$A_{[2]} = \lambda_1^2 + 2\lambda_1\lambda_2$$
$$A_{[3]} = \lambda_1^2\lambda_2$$

Because λ_1 has a multiplicity two, so $P_A(\lambda_1) = 0$ and $P_A{}'(\lambda_1) = 0$, as:

$$3\lambda_1^2 - 2\,trace(A)\lambda_1^2 + A_{[2]} = 0$$

Such that:

$$3\lambda_1^2 - 2\,trace(A)\lambda_1^2 + A_{[2]} = (\lambda - \lambda_1)^2 + 2(\lambda - \lambda_1)(\lambda - \lambda_2).$$

Since the above form is a quadratic form and has one round root, λ_1, then the others must be rational. Note that when the quadratic form is solved, there are $k \in Z^+$ such that:

$$4\left(trace(A)\right)^2 - 12A_{[2]} = (2k)^2$$

When divided by 4 it becomes:

$$\left(trace(A)\right)^2 - 3A_{[2]} = k^2$$

(iv) Suppose the non-zero eigenvalues of A are denoted by $\lambda_1, \lambda_2, \lambda_3$. Since $trace(A) = 0$, then $\lambda_1 + \lambda_2 + \lambda_3 = 0$, so $\lambda_3 = -(\lambda_1 + \lambda_2)$.

Note that:

$$\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 = -A_{[2]}$$
$$\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2 = -A_{[3]}$$

By performing the multiplication operation λ_1 on the above equation, we get:

$$\lambda_1^3 + \lambda_1^2\lambda_2 + \lambda_1\lambda_2^2 = -A_{[2]}\lambda_1$$

In other hands

$$\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2 = -A_{[3]}$$

So, it is obtained,

$$\lambda_1^3 + A_{[2]}\lambda_1 - A_{[3]} = 0$$

To solve the above equation in cubic form, the Cardano formula is used, in order to obtain

$$\sqrt[3]{\frac{A_{[3]}}{2} + \sqrt{\frac{\left(A_{[3]}\right)^2}{4} + \frac{\left(A_{[3]}\right)^3}{27}}} + \sqrt[3]{\frac{A_{[3]}}{2} - \sqrt{\frac{\left(A_{[3]}\right)^2}{4} + \frac{\left(A_{[3]}\right)^3}{27}}} = \sqrt[3]{\frac{A_{[3]}}{2} + k} + \sqrt[3]{\frac{A_{[3]}}{2} - k}$$
$$= m + n$$

Note that:

$$m \cdot n = \sqrt[3]{\frac{A_{[3]}}{2} + k} \cdot \sqrt[3]{\frac{A_{[3]}}{2} - k} = \sqrt[3]{\frac{\left(A_{[3]}\right)^2}{4} - k^2} = \sqrt[3]{\frac{\left(A_{[3]}\right)^2}{4} - \left(\frac{\left(A_{[3]}\right)^2}{4} - \frac{\left(A_{[3]}\right)^3}{27}\right)}$$
$$= -\frac{A_{[2]}}{3}$$
$$(m+n)^3 = m^3 + 3m^2n + 3mn^2 + n^3 = \left(\frac{A_{[3]}}{2} + k\right) + 3mn(m+n) + \left(\frac{A_{[3]}}{2} - k\right)$$
$$= \left(\frac{A_{[3]}}{2} + k\right) - \frac{A_{[2]}}{3} \cdot (m+n) + \left(\frac{A_{[3]}}{2} - k\right) = -\frac{A_{[2]}}{3} \cdot (m+n) + A_{[3]}.$$

Theorem 5. Let $A \in M_n(Z)$ with rank 4. If eigenvalues are given by $\lambda_1, \lambda_2 = 1$, and there exists $k \in Z$ such that:

$$\left(2 - trace(A)\right)^2 - 4A_{[4]} = k^2$$

then the other eigenvalues are integers.

Proof.

Let $A \in M_n(Z)$ with rank 4, then the characteristic polynomial:

$$P_A(\lambda) = \lambda^n - trace(A)\lambda^{n-1} + A_{[2]}\lambda^{n-2} - A_{[3]}\lambda^{n-3} + A_{[4]}$$
$$= \lambda^{n-4}(\lambda^4 - trace(A)\lambda^3 + A_{[2]}\lambda^2 - A_{[3]}\lambda + A_{[4]})$$

Since $\lambda_1, \lambda_2 = 1$

so, for $\lambda_1 = 1$, it is obtained:

$$1 - trace(A) + A_{[2]} - A_{[3]} + A_{[4]} = 0$$
$$P_A(\lambda) = \lambda^{n-4}(\lambda^4 - trace(A)\lambda^3 + A_{[2]}\lambda^2 - A_{[3]}\lambda + A_{[4]})$$

When it is factored, then it will be obtained,

$$P_A(\lambda) = \lambda^{n-4}(\lambda - 1)(\lambda^3 + (1 - trace(A))\lambda^3 + (A_{[4]} - A_{[3]})\lambda^2 - A_{[4]})$$

The form above still contains the power of three, it is difficult to find the eigenvalues, because one more eigenvalue is given, so we substitute it into:

For $\lambda_2 = 1$, it is obtained:

$$1 + 1 - trace(A) + A_{[4]} - A_{[3]} - A_{[4]} = 0$$
$$P_A(\lambda) = \lambda^{n-4}(\lambda - 1)(\lambda^3 + (1 - trace(A))\lambda^3 + (A_{[4]} - A_{[3]})\lambda^2 - A_{[4]})$$

When it is factored, then it will be obtained,

$$P_A(\lambda) = \lambda^{n-4}(\lambda - 1)(\lambda - 1)(\lambda^2 + (1 + 1 - trace(A))\lambda^3 + A_{[4]})$$

Because the above form produces a quadratic form, then with the abc formula, we get:

$$\lambda_{3,4} = \frac{-(2 - trace(A)) \pm \sqrt{\left(2 - trace(A)\right)^2 - 4A_{[4]}}}{2}$$

Based on the rank 2 case, there are $k \in Z$ such that:

$$\left(2 - trace(A)\right)^2 - 4A_{[4]} = k^2$$

Because of the difference is $\left(2 - trace(A)\right)^2 - k^2 = 4A_{[4]}$

So, since $\lambda_{3,4} = \frac{-(2 - trace(A)) \pm k}{2}$ such that $2 - trace(A)$ and k are both of even and odd, it means $\lambda_{3,4} \in Z$.

Theorem 6. Let $A \in M_n(Z)$ with rank 5. If eigenvalues are given by $\lambda_1, \lambda_2, \lambda_3 = 1$, and there exists $k \in Z$ such that:

$$\left(3 - trace(A)\right)^2 - 4A_{[5]} = k^2$$

then the other eigenvalues are integers.

Proof.

Let $A \in M_n(Z)$ with rank 5, then the characteristic polynomial

$$P_A(\lambda) = \lambda^n - trace(A)\lambda^{n-1} + A_{[2]}\lambda^{n-2} - A_{[3]}\lambda^{n-3} + A_{[4]}\lambda^{n-4} - A_{[5]}\lambda^{n-5}$$
$$= \lambda^{n-5}(\lambda^5 - trace(A)\lambda^4 + A_{[2]}\lambda^3 - A_{[3]}\lambda^2 + A_{[4]}\lambda - A_{[5]})$$

Since $\lambda_1, \lambda_2, \lambda_3 = 1$

So, for $\lambda_1 = 1$, it is obtained:

$$1 - trace(A) + A_{[2]} - A_{[3]} + A_{[4]} - A_{[5]} = 0$$
$$P_A(\lambda) = \lambda^{n-5}(\lambda^5 - trace(A)\lambda^4 + A_{[2]}\lambda^3 - A_{[3]}\lambda^2 + A_{[4]}\lambda - A_{[5]})$$

When it is factored, then it will be obtained,

$$P_A(\lambda) = \lambda^{n-4}(\lambda - 1)(\lambda^4 + \left(1 - trace(A)\right)\lambda^3 + \left(1 - trace(A) + A_{[2]}\right)\lambda^2 + \left(A_{[5]} - A_{[4]}\right)\lambda + A_{[5]}$$

The form above still contains the fourth power, it is difficult to find the eigenvalues, because one more eigenvalue is given, so we substitute it into:

For $\lambda_2 = 1$, it is obtained:

$$1 + 1 - trace(A) + 1 - trace(A) + A_{[2]} + A_{[5]} - A_{[4]} + A_{[5]} = 0$$
$$P_A(\lambda) = \lambda^{n-5}(\lambda - 1)(\lambda^4 + \left(1 - trace(A)\right)\lambda^3 + \left(1 - trace(A) + A_{[2]}\right)\lambda^2 + \left(A_{[5]} - A_{[4]}\right)\lambda + A_{[5]}$$

When it is factored, then it will be obtained,

$$P_A(\lambda) = \lambda^{n-5}(\lambda - 1)(\lambda - 1)\left(\lambda^3 + \left(1 + 1 - trace(A)\right)\lambda^2 + (A_{[4]} - 2A_{[5]}\right)\lambda - A_{[5]})$$

The form above still contains the power of three, it is difficult to find the eigenvalues, because one more eigenvalue is given, so we substitute it into:

For $\lambda_3 = 1$, it is obtained:

$$1 + 1 + 1 - trace(A) + 1 - trace(A) + A_{[4]} - 2A_{[5]} - A_{[5]} = 0$$
$$P_A(\lambda) = \lambda^{n-5}(\lambda - 1)(\lambda - 1)\left(\lambda^3 + \left(1 + 1 - trace(A)\right)\lambda^2 + (A_{[4]} - 2A_{[5]}\right)\lambda - A_{[5]})$$

When it is factored, then it will be obtained,

$$P_A(\lambda) = \lambda^{n-5}(\lambda - 1)(\lambda - 1)(\lambda - 1)\left(\lambda^2 + \left(1 + 1 + 1 - trace(A)\right)\lambda + A_{[5]}\right)$$

Because the above form produces a quadratic form, then with the abc formula, we get:

$$\lambda_{4,5} = \frac{-(3 - trace(A)) \pm \sqrt{\left(3 - trace(A)\right)^2 - 4A_{[5]}}}{2}$$

Based on the rank 2 case, there are $k \in Z$ such that:

$$\left(3 - trace(A)\right)^2 - 4A_{[5]} = k^2$$

Because of the difference is $\left(3 - trace(A)\right)^2 - k^2 = 4A_{[5]}$

Then $\lambda_{4,5} = \frac{-\left(3 - trace(A)\right) \pm k}{2}$ so $3 - trace(A)$ and k are both of even and odd, it means $\lambda_{4,5} \in Z$.

4. Conclusion

Based on the results of the research described in the previous chapter, it was obtained the following conclusions.

Characteristics of an integer symmetric matrix:

1. If $A \in M_n(Z)$ with rank 1, so A has integer eigenvalue.
2. If $A \in M_n(Z)$ with rank 2, so A has integer eigenvalue if and only if there exist two integers $m, n \in Z$ such that $trace(A) = m + n$ dan $A_{[2]} = m * n$, where $A_{[2]}$ is the sum of determinants of all 2nd order principal minors of A.

3. Let $A \in M_n(Z)$ with rank 3. If one of the following cases hold, then A has integer eigenvalues.

   (i) One of the eigenvalues of A is 1 or -1 and there exists a positive integer $k \in Z$ such that:
   $$\left[A_{[3]} - A_{[2]}\right]^2 - 4A_{[3]} = k^2$$
   (ii) All nonzero eigenvalues of A are the same and
   $$A_{[2]} = \frac{trace(A)^2}{3}, \qquad A_{[3]} = \frac{trace(A)^3}{27}$$
   (iii) One of the nonzero eigenvalues of A has multiplicity two and there exists a positive integer $k \in Z^+$such that: $trace(A)^2 - 3A_{[2]} = k^2$
   (iv) $Trace(A) = 0$ and there exists a positive integer $k \in Z^+$ and $m, n \in Z$ such that:
   $$k = \sqrt{\frac{\left(A_{[3]}\right)^2}{4} + \frac{\left(A_{[2]}\right)^3}{27}}$$
   $$m^3 = \frac{A_{[3]}}{2} + k, \quad n^3 = \frac{A_{[3]}}{2} - k$$
   In fact, one of eigenvalues $m + n$.
4. Let $A \in M_n(Z)$ with rank 4. If eigenvalues are given by $\lambda_1, \lambda_2 = 1$, and there exists $k \in Z$ such that:
   $$\left(2 - trace(A)\right)^2 - 4A_{[4]} = k^2$$
   then the other eigenvalues are integers.
5. Let $A \in M_n(Z)$ with rank 5. If eigenvalues are given by $\lambda_1, \lambda_2, \lambda_3 = 1$, and there exists $k \in Z$ such that:
   $$\left(3 - trace(A)\right)^2 - 4A_{[5]} = k^2$$
   then the other eigenvalues are integers.

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