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GRACEFUL LABELING OF SOME JOIN GRAPHS

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ABSTRACT

Let $G := (V,E)$ be a *graph* with a non-empty vertex set V and edge set E. We call G a $(p,q)-graph$ if $|V(G)| = p$ and $|E(G)| = q$. A *graceful labeling* of G is an injection $f : V(G) \to \{0,1,2,...,q\}$ such that the induced mapping f^*, defined by $f^*(uv) = |f(u) - f(v)|$ for each edge uv in G, is a bijection from $E(G)$ onto $\{1,2,3,...,q\}$. G which admits a graceful labeling is called *graceful graph*. Let G and H be two disjoint graphs. The *join* of G and H, denoted by $G+H$, is the union of G and H which is completed with edges joining each vertex in G to each vertex in H. If G and H are $(m,s)-graph$ and $(n,t)-graph$ respectively, then the join of both graphs will have size $mn+s+t$. In this paper, we will present two families of join graphs which have graceful labelings: $P(m,s)+I(n,t)$ and $P(m,s)+P(n,t)$.

Keyword: Graceful graph, graceful labeling, join graphs.

1 Introduction

Let $G := (V,E)$ be a *graph* with a non-empty vertex set V and edge set E. We call G n $(m,n)-graph$ if $|V(G)| = m$ and $|E(G)| = n$. A *graceful labeling* of G is an injection $f : V(G) \to \{0,1,2,...,q\}$ such that the induced mapping f^*, defined by $f^*(uv) = |f(u) - f(v)|$ for each edge uv in G, is a bijection from $E(G)$ onto $\{1,2,3,...,q\}$. G which admits a graceful labeling is called *graceful graph*. Since decades, there have been many publications regarding graceful graphs. In 1966, Rosa *et al.* [1] invented some properties of graceful labeling for trees. Almost 50 years later, Poomsa [2] has shown that some classes of spider graphs are graceful. Suparta and Ariawan [3] gave methods for constructing graceful classes of caterpillars, lobsters, and uniform trees that generalize results in [4]. Also, Suparta and Ariawan in [5] provide two methods for expanding graceful trees from certain graceful trees. For more survey of graph labelings, especially graceful graph, we refer to Gallian [6].

Let G and H be two disjoint graphs. The join of G and H, denoted by $G+H$, is the graph obtained from the union of G and H, and by joining each vertex in G to each vertex in H. If G and H are $(m,s)-graph$ and $(n,t)-graph$ respectively, then the join of both graphs will have size $mn+s+t$. In 2015, Koh, Phoon, & Soh [7] presented some new families of graceful join

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graphs. Throughout this paper, we shall discuss the gracefulness of other types of join graphs which are presented as open problems in [7]. Non-connected graph $P(m,s)$ is the union of path graph with size s and null graph with order $m-(s+1)$. While $I(n,t)$ is a non-connected graph with order n and with t disjoint edge(s), $n \geq 2t+1$. The join graphs that we will discuss in this paper are $P(m,s)+I(n,t)$ and $P(m,s)+P(n,t)$.

2 Main Result

As the results of we want to describe in this paper are some partial solutions of the open problems introduced in [7]. We will discuss the graceful labeling of join graphs $P(m,s)+I(n,t)$ and $P(m,s)+P(n,t)$ for some pairs of (m,s) and (n,t).
In the next discussion, for the edge labels of edge set E will be denoted by $L(E)$.

2.1 The join $P(m,s)+I(n,t)$

Theorem 1. If $s,t \in \{1,2\}$, then the join $P(m,s)+I(n,t)$ is graceful for all $m \geq s+2$ and $n \geq 2t+1$.

Proof. Let V be the vertex set of the graph $P(m,s)+I(n,t)$. It is easy to see that the graph has order $m+n$ and size $mn+s+t$. The proof will be devided into 4 cases:

Case 1. For $P(m,1)+I(n,1)$ with $m,n \geq 3$

Define a function $f: V(G) \to \{0,1,2,...,mn+2\}$ and edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_1v_2\}$;
$E_2 = \{v_iv_{m+j} | 1 \leq i \leq m, 1 \leq j \leq n\}$;
$E_3 = \{v_{m+1}v_{m+2}\}$.

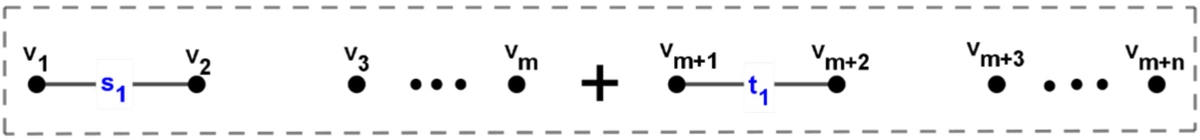

Figure 1: Join graph $P(m,1)+I(n,1)$

Here, we construct a vertex function

$$f(v_i) = \begin{cases} 0, & \text{if } i = 1, \\ (m+1-i)n+2, & \text{if } i = 2,3,...,m, \\ 1, & \text{if } i = m+1, \\ 2+j+(m-1)n, & \text{if } i = m+j; j = 2,3,...,n. \end{cases} \quad (1)$$

It is easy to verify that this function f defines an injection. This can be seen for instance by listing the image values of $v_1, v_2, \ldots, v_{m+n}$ under labeling function f as follows.

$$\begin{aligned}\{f(v_i) : i = 1,2,\ldots,m+n\} = &\ \{f(v_1)\} \cup \{f(v_i) : i = 2,\ldots,m\}\\ &\cup \{f(v_{m+1}) \cup \{f(v_i) : i = m+2,\ldots,m+n\}\\ = &\ \{0\} \cup \{(m-1)n+2,(m-2)n+2,\ldots,2n+2,n+2\} \cup \{1\}\\ &\cup \{(m-1)n+4,(m-1)n+5,\ldots,mn+2\}\\ = &\ \{0\} \cup \{1\} \cup \{n+2,2n+2,\ldots,(m-1)n+2\}\\ &\cup \{(m-1)n+4,(m-1)n+5,\ldots,mn+2\}.\end{aligned}$$

From the above listing, we can see immediately that each vertex gets distinct label, and moreover, the range of label set is in $\{0,1,\ldots,mn+2\}$. Thus we may conclud that f is injection from the vertex set into $\{0,1,\ldots,|E| = mn+2\}$.

Now we will see that the function (1) induces a bijection f^* from E onto $\{1,2,3,...,mn+2\}$.

$$\begin{aligned}L(E_1) &= \{f^*(v_1v_2)\} = \{mn-(n-2)\};\\ L(E_2) &= \{f^*(v_iv_{m+j}) | 1 \le i \le m, 1 \le j \le n\}\\ &= \{1,2,3,...,mn-n,mn-(n-1),mn-(n-4),...,mn+1,mn+2\};\\ L(E_3) &= \{f^*(v_{m+1}v_{m+2})\} = \{mn-(n-3)\}.\end{aligned}$$

Now we have:

$$\begin{aligned}L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\}\\ &= \{1,2,3,...,mn-(n-1),mn-(n-2),mn-(n-3),...,mn+1mn+2\}\\ &= \{1,2,3,...,mn+2\}.\end{aligned}$$

This says that f^* is bijection from E onto $\{1,2,3,...,mn+2\}$. Thus we show already that join graph $P(m,1)+I(n,1)$, with $m,n \ge 3$, is graceful graph.$\diamond$

Case 2. For $P(m,2)+I(n,1)$ with $m \ge 4$ and $n \ge 3$

First, define $f : V(G) \to \{0,1,2,...,mn+3\}$ as follows:

$$f(v_i) = \begin{cases} n+2, & \text{if } i = 1,\\ 1, & \text{if } i = 2,\\ mn+3-n, & \text{if } i = 3,\\ n(i-2)+2, & \text{if } i = 4,5,...,m,\\ 0, & \text{if } i = m+1,\\ mn-j+5, & \text{if } i = m+j; j = 2,3,...,n. \end{cases} \quad (2)$$

For concluding that f in Eq.(2) is an injection, we may argue using the similar arguments as we did for vertex label function in (1).

In the sequel, for the sake of efficiency, we will leave the proof of injection for any vertex label function.

Then, we have edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with

$E_1 = \{v_iv_{i+1} | 1 \le i \le 2\}$;

$E_2 = \{v_iv_{m+j} | 1 \le i \le m, 1 \le j \le n\}$;

$E_3 = \{v_{m+1}v_{m+2}\}$.

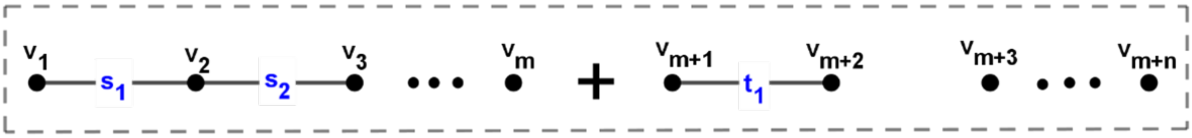

Figure 2: Join graph $P(m,2)+I(n,1)$

The following observations imply that the function (2) induces a bijection f^* from E onto $\{1,2,3,...,mn+3\}$.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3)\} \\ &= \{n+1, mn-(n-2)\}; \\ L(E_2) &= \{f^*(v_iv_{m+j})|1 \le i \le m, 1 \le j \le n\} \\ &= \{1,2,3,...,n,n+2,...,mn-(n-1),mn-(n-3),...,mn+1,mn+2\}; \\ L(E_3) &= \{f^*(v_{m+1}v_{m+2})\} = \{mn+3\}. \end{aligned}$$

we already have:

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,n+1,n+2,...,mn-(n-2),mn-(n-3),...,mn+2,mn+3\} \\ &= \{1,2,3,...,mn+3\}. \end{aligned}$$

Therefore, we can see that all edges receive distinct labels from $\{1,2,3,...,mn+3\}$. It follow that $P(m,2)+I(n,1)$, with $m \ge 4$ and $n \ge 3$, is graceful graph.$\diamond$

Case 3. For $P(m,1)+I(n,2)$ with $m \ge 3$ and $n \ge 5$

Here, define a function $f: V(G) \to \{0,1,2,...,mn+3\}$ and edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with

$E_1 = \{v_1v_2\}$;

$E_2 = \{v_iv_{m+j}|1 \le i \le m, 1 \le j \le n\}$;

$E_3 = \{v_{m+i-1}v_{m+i}|1 < i \le 4, i \text{ even}\}$.

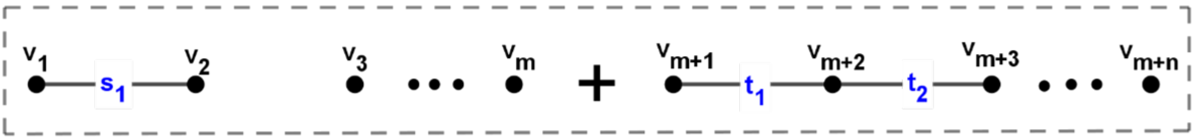

Figure 3: Join graph $P(m,1)+I(n,2)$

Now, we introduce a vertex function

$$f(v_i) = \begin{cases} 0, & \text{if } i=1, \\ mn+5-i, & \text{if } i=2,3,...,m, \\ 1, & \text{if } i=m+1, \\ (n-4+j)m+3, & \text{if } i=m+j; j=2,3, \\ (n+1-k)m+2, & \text{if } i=m+k; k=4,5,...,n-1, \\ m+1, & \text{if } i=m+n. \end{cases} \quad (3)$$

It can be shown directly that the function (3) is a graceful labeling for this case.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2)\} = \{mn+3\}; \\ L(E_2) &= \{f^*(v_iv_{m+j}) | 1 \le i \le m, 1 \le j \le n\}; \\ &= \{1,2,3,...,2m,2m+2,...,mn-(2m-1),mn-(2m-3),...,mn+2\}; \\ L(E_3) &= \{f^*(v_{m+1}v_{m+2}), f^*(v_{m+3}v_{m+4})\} \\ &= \{2m+1, mn-(2m-2)\}. \end{aligned}$$

We have:

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,2m+1,...,mn-(2m-2),mn-(2m-3),...,mn+2,mn+3\} \\ &= \{1,2,3,...,mn+3\}. \end{aligned}$$

Since all the adges of $P(m,1)+I(n,2)$, with $m \ge 3$ and $n \ge 5$, receive distinct labels from 1 to $mn+3$, the function (3) is graceful labeling and $P(m,1)+I(n,2)$ is a graceful graph.$\diamond$

Case 4. For $P(m,2)+I(n,2)$ with $m \ge 4$ and $n \ge 5$

We will define a function $f : V(G) \to \{0,1,2,...,mn+4\}$ as follows

$$f(v_i) = \begin{cases} 0, & \text{if } i = 1, \\ mn+4, & \text{if } i = 2, \\ 1, & \text{if } i = 3, \\ mn+3, & \text{if } i = 4, \\ i-3, & \text{if } i = 5,6,...,m, \\ 1+3m, & \text{if } i = m+1, \\ (j-1)m, & \text{if } i = m+j; j = 2,3, \\ (n+4-k)m+2, & \text{if } i = m+k; k = 4,5, \\ (n+4-l)m+1, & \text{if } i = m+l; l = 6,7,...,n. \end{cases} \tag{4}$$

Then the edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_iv_{i+1} | 1 \le i \le 2\}$;
$E_2 = \{v_iv_{m+j} | 1 \le i \le m, 1 \le j \le n\}$;
$E_3 = \{v_{m+i-1}v_{m+i} | 1 < i \le 4, i \text{ even}\}$;

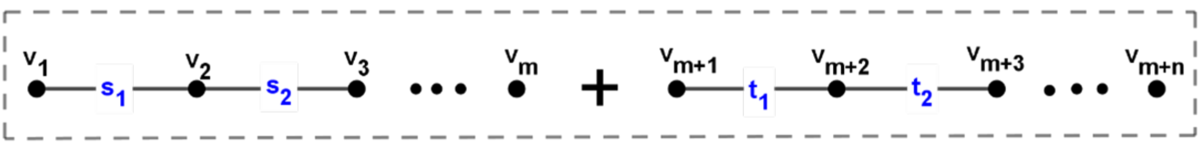

Figure 4: Join graph $P(m,2)+I(n,2)$

It is easy to see that the function (4) induces a bijection f^* from E onto $\{1,2,3,...,mn+4\}$

as we describe below.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3)\} \\ &= \{mn+3, mn+4\}; \\ L(E_2) &= \{f^*(v_iv_{m+j}) | 1 \leq i \leq m, 1 \leq j \leq n\} \\ &= \{1,2,3,...,2m,2m+2,...,mn-(2m-1),mn-(2m-3),...,mn+2\}; \\ L(E_3) &= \{f^*(v_{m+1}v_{m+2}), f^*(v_{m+3}v_{m+4})\} \\ &= \{2m+1, mn-(2m-2)\}. \end{aligned}$$

Then, we have:

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,2m+1,2m+2,...,mn-(2m-2),...,mn+2,mn+3,mn+4\} \\ &= \{1,2,3,...,mn+4\}. \end{aligned}$$

We clearly shown that $P(m,2)+I(n,2)$, with $m \geq 4$ and $n \geq 5$, is graceful graph.$\diamond$

Theorem 2. Join graph $P(m,3)+I(n,2)$ is graceful for all $m \geq n \geq 5$.

Proof. Let V be the vertex set of the graph $P(m,3)+I(n,2)$, it is easy to see that the graph has size $mn+3+2 = mn+5$. Define a function $f : V(G) \to \{0,1,2,...,mn+5\}$ and edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_iv_{i+1} | 1 \leq i \leq 3\}$;
$E_2 = \{v_iv_{m+j} | 1 \leq i \leq m, 1 \leq j \leq n\}$;
$E_3 = \{v_{m+i-1}v_{m+i} | 1 < i \leq 4, i \text{ even}\}$.

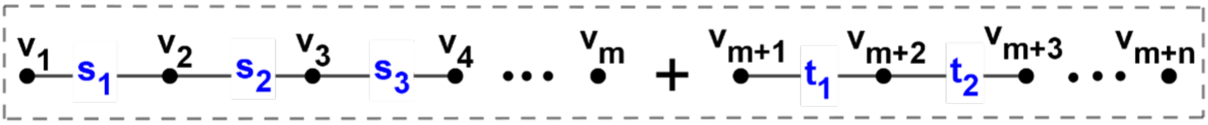

Figure 5: Join graph $P(m,3)+I(n,2)$

Now we will set a vertex function

$$f(v_i) = \begin{cases} 0, & \text{if } i = 1, \\ mn+5, & \text{if } i = 2, \\ 1, & \text{if } i = 3, \\ mn+4, & \text{if } i = 4, \\ m-1, & \text{if } i = 5, \\ i-4, & \text{if } i = 6,7,...,m, \\ m, & \text{if } i = m+1, \\ m(n-1)+2, & \text{if } i = m+2, \\ mn+2, & \text{if } i = m+3, \\ (n+2-j)m+1, & \text{if } i = m+j; j = 4,5,...,n-1, \\ 2m, & \text{if } i = m+n. \end{cases} \quad (5)$$

It is easy to observe that the function (5) induces a bijection f^* from E onto $\{1,2,3,...,mn+5\}$ as follows

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4)\} \\ &= \{mn+3, mn+4, mn+5\}; \\ L(E_2) &= \{f^*(v_iv_{m+j}) | 1 \le i \le m, 1 \le j \le n\} \\ &= \{1,2,3,...,m-2,m-1,...,2m-2,2m,...,mn-(2m-1),...,mn+1\}; \\ L(E_3) &= \{f^*(v_{m+1}v_{m+2}), f^*(v_{m+3}v_{m+4})\} = \{2m+1, mn+2\}. \end{aligned}$$

We have

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,2m-2,2m-1,...,mn+1,mn+2,mn+3,mn+4,mn+5\} \\ &= \{1,2,3,...,mn+5\}. \end{aligned}$$

The proof says that f^* is a bijection from E onto $\{1,2,3,...,mn+5\}$. Thus we have proven that $P(m,3)+I(n,2)$, with $m \ge 5$ and $n \ge 5$, is graceful graph.◇

2.2 The Join $P(m,s)+P(n,t)$

In this section, we will discuss the gracefulness of join graph $P(m,s)+P(n,t)$ with $m \ge s+2$ and $n \ge t+2$. In certain case, when $t=1$, graph $P(m,s)+P(n,t)$ will be equivalent to graph $P(m,s)+I(n,t)$. Consequently, both types of graphs can have the same labeling function.

Theorem 3. If $m \in \{5,6\}$, then the join graph $P(m,3)+I(n,1)$ is graceful for all $n \ge 3$.

Note: The join $P(m,3)+P(n,1)$ is equivalent with the join $P(m,3)+I(n,1)$. So this theorem can be used for both types of graph.

Proof. Let V be the vertex set of the graph $P(m,3)+P(n,1)$, it is easy to see that the graph has order $m+n$ and size $mn+4$.

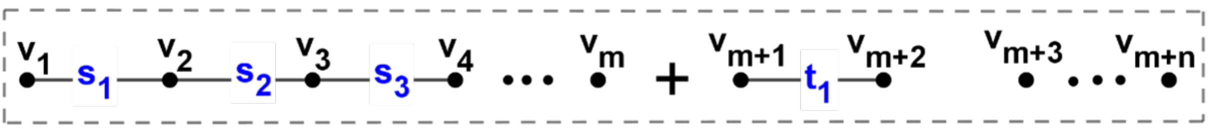

Figure 6: Join graph $P(m,3)+P(n,1)$

The proof will be devided into 2 cases:
Case 1. For $m=5$

First, we introduce a function $f : V(G) \to \{0,1,2,...,5n+4\}$ as follows

$$f(v_i) = \begin{cases} 1+n, & \text{if } i=1, \\ 4+3n, & \text{if } i=2, \\ 1, & \text{if } i=3, \\ 2(1+n)(i-3), & \text{if } i=4,5 \\ 0, & \text{if } i=6, \\ 5n-j+6, & \text{if } i=5+j; j=2,3,...,n. \end{cases} \quad (6)$$

Then, we have edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_i v_{i+1} | 1 \le i \le 3\}$;
$E_2 = \{v_i v_{5+j} | 1 \le i \le 5, 1 \le j \le n\}$;
$E_3 = \{v_6 v_7\}$.

It is easy to observe that the function (6) induces a bijection f^* from each edge label onto $\{1,2,3,...,5n+4\}$ as we see below.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4)\} \\ &= \{2n+1, 2n+5, 3n+3\}; \\ L(E_2) &= \{f^*(v_iv_{5+j}) | 1 \le i \le 5, 1 \le j \le n\} \\ &= \{1,2,3,...,2n,2n+2,...,2n+4,2n+6,...,3n+2,3n+4,...,5n+3\}; \\ L(E_3) &= \{f^*(v_6v_7)\} = \{5n+4\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,2n+1,2n+2...,2n+5,...,3n+3,3n+4,...,5n+3,5n+4\} \\ &= \{1,2,3,...,5n+4\}. \end{aligned}$$

Since f^* is a bijection from E onto $\{1,2,3,...,5n+4\}$, then $P(m,3)+P(n,1)$, with $m=5$ and $n \le 3$, has graceful labeling.$\diamond$

Case 2. For $m=6$

First, we define a function $f : V(G) \to \{1,2,3,...,6n+4\}$ and edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_i v_{i+1} | 1 \le i \le 3\}$;
$E_2 = \{v_i v_{6+j} | 1 \le i \le 6, 1 \le j \le n\}$;
$E_3 = \{v_7 v_8\}$;

Then, we generate a vertex function

$$f(v_i) = \begin{cases} 5n+4, & \text{if } i=1, \\ i(n+1), & \text{if } i=2,4, \\ 1, & \text{if } i=3, \\ 3n+3, & \text{if } i=5, \\ n+1, & \text{if } i=6, \\ 0, & \text{if } i=7, \\ 6n-j+6, & \text{if } i=6+j; j=2,3,...,n. \end{cases} \quad (7)$$

The following observations imply that the function (7) induces f^* from each edge label onto $\{1,2,3,...,6n+4\}$.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4)\} \\ &= \{2n+1, 3n+2, 4n+3\}; \\ L(E_2) &= \{f^*(v_iv_{6+j}) | 1 \le i \le 6, 1 \le j \le n\} \\ &= \{1,2,3,...,2n, 2n+2,...,3n+1, 3n+3,...,4n+2, 4n+4,...,6n+3\}; \\ L(E_3) &= \{f^*(v_6v_7)\} = \{6n+4\}. \end{aligned}$$

We get

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,2n, 2n+1,...,3n+2, 3n+3,...,4n+3, 4n+4...,6n+3, 6n+4\} \\ &= \{1,2,3,...,6n+4\}. \end{aligned}$$

This says that all edges receive distinct label from 1 to $6n+4$, thus we show already that $P(m,3)+P(n,1)$, with $m=6$ and $n \ge 3$, is graceful graph.$\diamond$

Theorem 4. The join $P(m,3)+P(n,2)$ is graceful for all $m \ge 5$ dan $n \ge 4$.

Proof. Let V be the vertex of the graph $P(m,3)+P(n,2)$, we can see that the graph has order $m+n$ and size $mn+3+2 = mn+5$. Now, define a function $f : V(G) \to \{0,1,2,...,mn+5\}$ as follows:

$$f(v_i) = \begin{cases} 0, & \text{if } i = 1, \\ mn+5, & \text{if } i = 2, \\ 1, & \text{if } i = 3, \\ mn+4, & \text{if } i = 4, \\ i-3, & \text{if } i = 5,6,...,m-1, \\ m-1, & \text{if } i = m, \\ m, & \text{if } i = m+1, \\ mn+2, & \text{if } i = m+2 \\ (n-j+2)m+1, & \text{if } i = m+j; j = 3,4,...,n. \end{cases} \quad (8)$$

Also, we have edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_iv_{i+1} | 1 \le i \le 3\}$
$E_2 = \{v_iv_{m+j} | 1 \le i \le m, 1 \le j \le n\}$
$E_3 = \{v_{m+i}v_{m+i+1} | 1 < i \le 2\}$

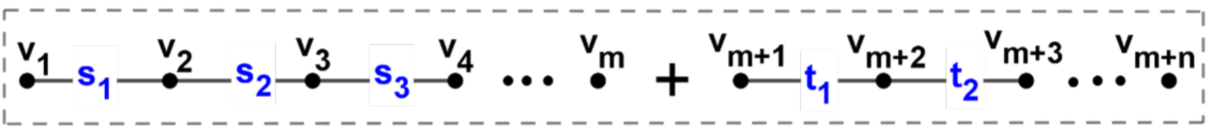

Figure 7: Join graph $P(m,3)+P(n,2)$

Futhermore, we will see that the function (8) can induces a bijection f^* from each edge onto $\{1,2,3,...,mn+5\}$.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4)\} \\ &= \{mn+3, mn+4, mn+5\}; \\ L(E_2) &= \{f^*(v_iv_{m+j}) | 1 \leq i \leq m, 1 \leq j \leq n\} \\ &= \{1,2,3,...,m-4, m-2,...,mn-(m-1), mn-(m-3),...,mn+2\}; \\ L(E_3) &= \{f^*(v_{m+1}v_{m+2}), f^*(v_{m+2}v_{m+3})\} \\ &= \{m-3, mn-(m-2)\}. \end{aligned}$$

We already have

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,m-3,...,mn-(m-2),...,mn+2, mn+3, mn+4, mn+5\} \\ &= \{1,2,3,...,mn+5\}. \end{aligned}$$

We have seen that f^* is a bijection from E onto $\{1,2,3,...,mn+5\}$. Thus, $P(m,3)+P(n,2)$, with $m \geq 5$ and $n \geq 4$, is graceful graph.$\diamond$

Theorem 5. If $m \in \{6,7,8\}$, then the join graph $P(m,4)+P(n,1)$ is graceful for all $n \geq 3$.
Note: The join $P(m,4)+P(n,1)$ is equivalent with the join $P(m,4)+I(n,1)$. So this theorem can be used for both types of graph.

Proof. Let V be the vertex set of the graph $P(m,4)+P(n,1)$, we see that the graph has order $m+n$ and size $mn+4+1 = mn+5$.

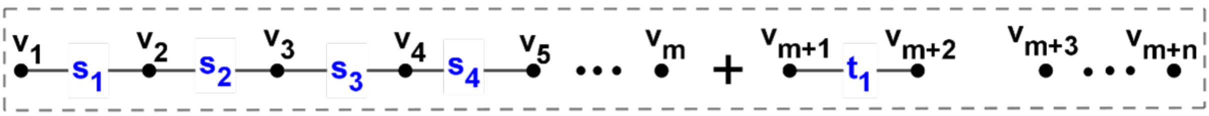

Figure 8: Join graph $P(m,4)+P(n,1)$

The proof will be devided by 3 cases:

Case 1. For $m = 6$

Define a function $f : V(G) \to \{0,1,2,...,6n+5\}$ and edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_iv_{i+1} | 1 \leq i \leq 4\}$;
$E_2 = \{v_iv_{6+j} | 1 \leq i \leq 6, 1 \leq j \leq n\}$;
$E_3 = \{v_7v_8\}$.

Now, we introduce a vertex function

$$f(v_i) = \begin{cases} 4n+4, & \text{if } i=1, \\ n+2, & \text{if } i=2, \\ 1, & \text{if } i=3, \\ 5n+5, & \text{if } i=4, \\ 2n+2, & \text{if } i=5, \\ 3n+4, & \text{if } i=6, \\ 0, & \text{if } i=7, \\ 6n-j+7, & \text{if } i=6+j; j=2,3,...,n. \end{cases} \quad (9)$$

We will observe that the function (9) is a graceful labeling.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4), f^*(v_4v_5)\} \\ &= \{n+1, 3n+2, 3n+3, 5n+4\}; \\ L(E_2) &= \{f^*(v_iv_{6+j}) | 1 \le i \le 6, 1 \le j \le n\}; \\ &= \{1,2,3,...,n,n+2,...,3n+1,3n+4,...,5n+3,5n+5,...,6n+4\}; \\ L(E_3) &= \{f^*(v_7v_8)\} = \{6n+5\}. \end{aligned}$$

We have

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,n,n+1,...,3n+2,3n+3,...,5n+4,5n+5,...,6n+4,6n+5\} \\ &= \{1,2,3,...,6n+5\}. \end{aligned}$$

This says that the function (9) induces a bijective f^* from E onto $\{1,2,3,...,6n+5\}$. Thus we already show that join graph $P(m,4)+P(n,1)$, with $m=6$ and $n \ge 3$, has a graceful labeling.$\diamond$

Case 2. For $m=7$

Now, we define a function $f: V(G) \to \{0,1,2,...,7n+5\}$ as follows:

$$f(v_i) = \begin{cases} n+2, & \text{if } i=1, \\ 1, & \text{if } i=2, \\ (-4i+18)n-3i+14, & \text{if } i=3,4, \\ (10-i)n+4, & \text{if } i=5,6, \\ 3n+3, & \text{if } i=7 \\ 0, & \text{if } i=8, \\ 7n+7-j, & \text{if } i=7+j; j=2,3,...,n. \end{cases} \quad (10)$$

Also, we have an edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_iv_{i+1} | 1 \le i \le 4\}$;
$E_2 = \{v_iv_{7+j} | 1 \le i \le 7, 1 \le j \le n\}$;
$E_3 = \{v_8v_9\}$;

We will see that the function (10) induces a bijection f^* from E onto $\{1,2,3,...,7n+5\}$.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4), f^*(v_4v_5)\} \\ &= \{n+1, 3n+2, 4n-3, 6n+4\}; \\ L(E_2) &= \{f^*(v_iv_{7+j}) | 1 \le i \le 7, 1 \le j \le n\} \\ &= \{1,...,n,n+2,...,3n+1,3n+3,...,4n-4,4n-2,...,6n+3,...,7n+4\}; \\ L(E_3) &= \{f^*(v_8v_9)\} = \{7n+5\}. \end{aligned}$$

Finally, we have

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,n+1,...,3n+2,...,4n-3,4n-3,...,6n+4,...,7n+4,7n+5\} \\ &= \{1,2,3,...,7n+5\}. \end{aligned}$$

We show already that the graph $P(m,4)+P(n,1)$, with $m=7$ and $n \geq 3$, has a graceful labeling.◇

Case 3. For $m=8$

We introduce a function $f: V(G) \to \{0,1,2,...,7n+5\}$ and edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_i v_{i+1} | 1 \leq i \leq 4\}$;
$E_2 = \{v_i v_{8+j} | 1 \leq i \leq 8, 1 \leq j \leq n\}$;
$E_3 = \{v_9 v_{10}\}$;

Now, we will construct a vertex function

$$f(v_i) = \begin{cases} n+2, & \text{if } i=1, \\ 1, & \text{if } i=2, \\ 22n-(5n+3)i+14, & \text{if } i=3,4, \\ in+4, & \text{if } i=5,6, \\ (11-i)n+3, & \text{if } i=7,8, \\ 0, & \text{if } i=9, \\ 8n+7-l, & \text{if } i=8+l; l=2,3,...,n. \end{cases} \tag{11}$$

We will observe that the function (11) can induces each edge labels so f^* is a bijection from E onto $\{1,2,3,...,8n+5\}$.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4), f^*(v_4v_5)\} \\ &= \{n+1, 3n+2, 5n+3, 7n+4\}; \\ L(E_2) &= \{f^*(v_i v_{8+j}) | 1 \leq i \leq 8, 1 \leq j \leq n\} \\ &= \{1,...,n,n+2,...,3n+1,3n+3,...,5n+2,5n+4,...,7n+3,...,8n+4\}; \\ L(E_3) &= \{f^*(v_8v_9)\} = \{8n+5\}. \end{aligned}$$

We already have

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3)\} \\ &= \{1,2,3,...,n,n+1,n+2,...,3n+2,...,5n+3,...,7n+4,...,8n+4,8n+5\} \\ &= \{1,2,3,...,8n+5\}. \end{aligned}$$

Therefore, we can see that f^* is a bijection from E onto $\{1,2,3,...,8n+5\}$. Thus, the graph $P(m,4)+P(n,1)$, with $m=8$ and $n \geq 3$, is graceful graph.◇

Theorem 6. If $m \in \{6,7,8\}$, then the join $P(m,4)+P(n,2)$ is graceful for all $n \geq 4$.

Proof. Let V be the vertex set of the graph $P(m,4)+P(n,2)$. It is easy to see that the graph has order $m+n$ and size $mn+4+2 = mn+6$.

Figure 9: Join graph $P(m,4)+P(n,2)$

The proof will be devided into 3 cases:

Case 1. For $m=6$

Here, define $f:V(G)\to\{0,1,2,...,6n+6\}$ as follows:

$$f(v_i)=\begin{cases}4n+3, & \text{if } i=1,\\ n, & \text{if } i=2,\\ (9-i)n+7-i, & \text{if } i=3,4\\ (i-3)n+1, & \text{if } i=5,6\\ 0, & \text{if } i=7,\\ 6n+6, & \text{if } i=8,\\ 1, & \text{if } i=9,\\ 6n+5, & \text{if } i=10,\\ k-3, & \text{if } i=6+k; k=5,6,...,n.\end{cases} \tag{12}$$

Also, we have an edge set $E=\{E_1\cup E_2\cup E_3\}$ with
$E_1=\{v_iv_{i+1}|1\le i\le 4\}$;
$E_2=\{v_iv_{6+j}|1\le i\le 6, 1\le j\le n\}$;
$E_3=\{v_{m+i}v_{m+i+1}|1\le 2\}$.

It can be shown directly that the function (12) is a graceful labeling for this case.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2),f^*(v_2v_3),f^*(v_3v_4),f^*(v_4v_5)\}\\ &= \{n+1,3n+2,3n+3,5n+4\};\\ L(E_2) &= \{f^*(v_iv_{6+j})|1\le i\le 6,1\le j\le n\}\\ &= \{1,2,3,...,n,n+2,...,3n+1,3n+4,...,5n+3,5n+5,...,6n+4\};\\ L(E_3) &= \{f^*(v_7v_8),f^*(v_8v_9)\}=\{6n+5,6n+6\}.\end{aligned}$$

We already have

$$\begin{aligned} L(E) &= \{L(E_1)\cup L(E_2)\cup L(E_3)\}\\ &= \{1,2,3,...,n+1,...,3n+2,3n+3,...,5n+4,...,6n+4,6n+5,6n+6\}\\ &= \{1,2,3,...,6n+6\}.\end{aligned}$$

Since all the edges of graph $P(m,4)+P(n,2)$, with $m=6$ and $n\ge 4$, receive distinct labels from 1 to $6n+6$, we show already that $P(m,4)+P(n,2)$ is graceful graph.$\diamond$

Case 2. For $m=7$

Define a function $f:V(G)\to\{0,1,2,...,7n+6\}$ and edge set $E=\{E_1\cup E_2\cup E_3\}$ with
$E_1=\{v_iv_{i+1}|1\le i\le 4\}$;
$E_2=\{v_iv_{7+j}|1\le i\le 7,1\le j\le n\}$;

$E_3 = \{v_{7+i}v_{7+i+1} | 1 \leq i \leq 2\}$.

Then, we generate a vertex function as follows:

$$f(v_i) = \begin{cases} 5n+3, & \text{if } i = 1, \\ n, & \text{if } i = 2, \\ (10-i)n+7-i, & \text{if } i = 3,4, \\ 3n+1 & \text{if } i = 5, \\ (2i-10)n-5+i, & \text{if } i = 6,7, \\ 0, & \text{if } i = 8, \\ 7n+6, & \text{if } i = 9, \\ 1, & \text{if } i = 10, \\ 7n+5, & \text{if } i = 11, \\ k-3, & \text{if } i = 7+k; k = 5,6...,n. \end{cases} \tag{13}$$

It is easy to see that the function (13) induces a bijection f^* from E onto $\{1,2,3,...,7n+6\}$ as we describe below.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4), f^*(v_4v_5)\} \\ &= \{n+1, 3n+2, 4n+3|, 6n+4\}; \\ L(E_2) &= \{f^*(v_iv_{7+j}) | 1 \leq i \leq 7, 1 \leq j \leq n\} \\ &= \{1,...,n,n+2,...,3n+1,3n+3,...,4n+2,4n+4,...,6n+3,...,7n+4\}; \\ L(E_3) &= \{f^*(v_8v_9), f^*(v_9v_{10})\} = \{7n+5, 7n+6\}. \end{aligned}$$

Now, we already have

$$\begin{aligned} L(E) &= \{L(E_1) \cup L(E_2) \cup L(E_3))\} \\ &= \{1,2,3,...,n+1,...,3n+2,...,4n+3,...,6n+4,...,7n+4,7n+5,7n+6\} \\ &= \{1,2,3,...,7n+6\}. \end{aligned}$$

We clearly shown that $P(m,4)+P(n,2)$, with $m = 7$ and $n \geq 4$, is graceful graph.$\diamond$

Case 3. For $m = 8$

We define a function $f : V(G) \to \{0,1,2,...,7n+5\}$ and edge set $E = \{E_1 \cup E_2 \cup E_3\}$ with
$E_1 = \{v_iv_{i+1} | 1 \leq i \leq 4\}$;
$E_2 = \{v_iv_{8+j} | 1 \leq i \leq 8, 1 \leq j \leq n\}$;
$E_3 = \{v_{8+i}v_{8+i+1} | 1 \leq i \leq 2\}$.

Here, we have a vertex labels function

$$f(v_i) = \begin{cases} (3n+2)i-1, & \text{if } i = 1,2, \\ n, & \text{if } i = 3, \\ (12-i)n+8-i, & \text{if } i = 4,5, \\ (2i-10)n-5+i, & \text{if } i = 6,7, \\ 5n+2, & \text{if } i = 8, \\ 0, & \text{if } i = 9, \\ 8n+6, & \text{if } i = 10, \\ 1, & \text{if } i = 11, \\ 8n+5, & \text{if } i = 12, \\ l-3, & \text{if } i = 8+l; l = 5,6,...,n. \end{cases} \tag{14}$$

Futhermore, we will observe that the function (14) induces a bijection f^* from E onto $\{1,2,3,...,8n+6\}$.

$$\begin{aligned} L(E_1) &= \{f^*(v_1v_2), f^*(v_2v_3), f^*(v_3v_4), f^*(v_4v_5)\} \\ &= \{n+1, 3n+2, 4n-3, 6n+4\}; \\ L(E_2) &= \{f^*(v_iv_{8+j}) | 1 \le i \le 8, 1 \le j \le n\} \\ &= \{1,...,n,n+2,...,3n+1,3n+3,...,5n+2,5n+4,...,7n+3,...,8n+4\}; \\ L(E_3) &= \{f^*(v_9v_{10}), f^*(v_{10}v_{11})\} = \{8n+5, 8n+6\}. \end{aligned}$$

Now, we get

$$\begin{aligned} E &= \{E_1 \cup E_2 \cup E_3\} \\ &= \{1,2,3,...,n+1,...,3n+2,...,5n+3,...,7n+3,...,8n+4,8n+5,8n+6\} \\ &= \{1,2,3,...,8n+6\}. \end{aligned}$$

Since f^* is a bijection from E onto $\{1,2,3,...,8n+6\}$, then $P(m,4)+P(n,2)$, with $m=8$ and $n \ge 4$, is graceful graph.$\diamond$

3 Conclusion

We have proved that the join graphs $P(m,s)+I(n,t)$ and $P(m,s)+P(n,t)$ are graceful for some pairs of (m,s) and (n,t). Doing observation on the remaining cases, regarding with the mentioned open problems as is stated in the beginning part, is till worth to do.

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