



NUMERICAL SOLUTION OF MIXED LINEAR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS BY MODIFIED BLOCK PULSE FUNCTIONS

AYYUBI AHMAD

Department of Mathematics, Faculty of Applied Science and Technology, Ahmad Dahlan University,
Yogyakarta, Indonesia

ayyubiahmad1@gmail.com

ABSTRACT

A numerical method based on modified block pulse functions is proposed for solving the mixed linear Volterra-Fredholm integral equations. An integration operational matrix of modified block pulse functions is obtained on interval $[0, T)$. A modified block pulse functions and their operational matrix of integration, the mixed linear Volterra-Fredholm integral equations can be reduced to a linear system of algebraic equations. The rate of convergence is $\mathcal{O}(h)$ and error analysis of the proposed method are discussed. Some examples are provided to show that the proposed method have a good degree of accuracy.

Keywords: Integration Operational Matrix, Itô Integral, Mixed Linear Volterra-Fredholm Integral Equations, ε MBPFs

1 Introduction

An integral equation is an equation in which an unknown function appears under one or more integration signs. The integral equations are found in various applied areas including mechanics, physics, and engineering. The subject of integral equations are used as mathematical models for many and varied physical situations. It also arise as representation formulas for the solutions of differential equations.

The mixed linear Volterra-Fredholm integral equation is a combination of disjoint linear Volterra and Fredholm integrals that appear in an integral equation. The mixed linear Volterra-Fredholm integral equations arise from parabolic boundary value problems, mathematical modelling of the spatio-temporal development of an epidemic, various physical, biological, and chemical applications [9, 10]. There are several techniques for approximating the solution such as designing neural network [1], linear programming problem [5], Taylor expansion method [2], homotopy perturbation method [4], Legendre collocation method [7], and Bernstein polynomials method [8].

In this paper, the modified block pulse functions is applied for solving the following mixed linear Volterra-Fredholm integral equations

$$X(t) = f(t) + \int_0^t \int_\alpha^\beta K(x, r)X(r) dr dx, \quad (1)$$

where $X(t)$ is the unknown function, $f(t)$ is analytic function and $K(x, r)$ is the kernels of L^2 functions.

2 Methodology

A set of ε modified block pulse functions $\psi_i(t)$, $i = 0, 1, \dots, m$ on the interval $[0, T)$ are defined as

$$\begin{aligned} \psi_0(t) &= \begin{cases} 1 & t \in [0, h - \varepsilon) = I_0, \\ 0 & \text{otherwise,} \end{cases} \\ \psi_i(t) &= \begin{cases} 1 & t \in [ih - \varepsilon, (i+1)h - \varepsilon) = I_i, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for $i = 1, 2, \dots, m-1$, and

$$\psi_m(t) = \begin{cases} 1 & t \in [T - \varepsilon, T) = I_m, \\ 0 & \text{otherwise.} \end{cases}$$

with a positive integer value for m and $h = \frac{T}{m}$.

Similar to BPFs, the important properties of ε MBPFs are as follows

- Disjointness:

$$\psi_i(t)\psi_j(t) = \begin{cases} \psi_i(t) & i = j, \\ 0 & i \neq j, \end{cases}$$

where $i, j = 0, \dots, m$.

- Orthogonality:

$$\int_0^T \psi_i(t)\psi_j(t) dt = h\delta_{ij}$$

where $i, j = 1, \dots, m-1$ and δ_{ij} is Kronecker delta.

- Completeness:

$$\int_0^T f^2(t) dt = \sum_{i=0}^{\infty} f_i^2 \|\psi_i(t)\|^2,$$

where

$$f_i = \frac{1}{\Delta(I_i)} \int_0^T f(t)\psi_i(t) dt \quad (2)$$

and $\Delta(I_i)$ is length of interval I_i .

Rewriting Eq. (2) in the vector form we have

$$f(t) \simeq \sum_{i=0}^m f_i \psi_i(t) = F^T \Psi(t) = \Psi^T(t) F,$$

in which

$$F = (f_0 \ f_1 \ \dots \ f_m)^T \quad \text{and} \quad \Psi(t) = (\psi_0(t) \ \psi_1(t) \ \dots \ \psi_m(t))^T.$$

Moreover, any two dimensional function $k(s, t) \in L^2([0, T_1] \times [0, T_2])$ can be expanded with respect to ε MBPFs such as

$$k(s, t) = \Psi^T(s) K \Phi(t) = \Phi^T(t) K^T \Psi(s),$$

where $\Psi(s)$ and $\Phi(t)$ are m_1 and m_2 dimensional ε MBPFs vectors respectively, and $K = (k_{ij})$, $i = 0, 1, \dots, m_1$, $j = 0, 1, \dots, m_2$ is the $m_1 \times m_2$ ε modified block pulse coefficient matrix with

$$k_{ij} = \frac{1}{\Delta(I_i)\Delta(I_j)} \int_0^{T_1} \int_0^{T_2} k(s, t) \Psi_i(s) \Phi_j(t) dt ds,$$

For convenience, we put $m_1 = m_2 = m$.

With defining $\Psi_{m+1}(t) = (\psi_0(t) \ \psi_1(t) \ \dots \ \psi_m(t))^T$, we have

$$\Psi_{m+1}(t)\Psi_{m+1}^T(t) = \begin{pmatrix} \psi_0(t) & 0 & \cdots & 0 \\ 0 & \psi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_m(t) \end{pmatrix}_{(m+1) \times (m+1)},$$

Furthermore,

$$\Psi_{m+1}^T(t)\Psi_{m+1}(t) = 1,$$

and

$$\Psi_{m+1}(t)\Psi_{m+1}^T(t)F = D_F\Psi_{m+1}(t)$$

where D_F usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_0 \ f_1 \ \dots \ f_m)^T$.

Similar to BPFs,

$$\int_0^t \Psi_{m+1}(s) ds \simeq Q\Psi_{m+1}(t),$$

where the integration operational matrix Q of ϵ MBPFs is given by

$$Q = \begin{pmatrix} \frac{h-\epsilon}{2} & h-\epsilon & \cdots & h-\epsilon \\ 0 & \frac{h}{2} & \cdots & h \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\epsilon}{2} \end{pmatrix}_{(m+1) \times (m+1)}.$$

So, the integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s) ds \simeq \int_0^t F^T\Psi_{m+1}(s) ds \simeq F^T Q\Psi_{m+1}(t).$$

3 Results

3.1 Solving Mixed Linear Volterra-Fredholm Integral Equations by Using ϵ Modified Block Pulse Functions

We consider following mixed linear Volterra-Fredholm integral equation

$$X(t) = f(t) + \int_0^t \int_\alpha^\beta K(x,r)X(r) dr dx, \quad (3)$$

We approximate functions $X(t)$, $f(t)$, and $K(x,r)$ by ϵ MBPFs:

$$X(t) \simeq W^T\Phi(t) = \Phi^T(t)W,$$

$$f(t) \simeq F^T\Phi(t) = \Phi^T(t)F,$$

$$K(x,r) \simeq \Phi^T(x)K\Phi(r) = \Phi^T(r)K^T\Phi(x).$$

In the above approximates, W and F are modified block pulse coefficients vector, and K are modified block pulse coefficients matrix.

Substituting above approximation in Eq. (3), we get

$$W^T\Phi(t) \simeq F^T\Phi(t) + \int_0^t \int_\alpha^\beta \Phi^T(x)K\Phi(r)\Phi^T(r)W dr dx$$

$$W^T \Phi(t) \simeq F^T \Phi(t) + \int_0^t \Phi^T(x) K \left(\int_\alpha^\beta \Phi(r) \Phi^T(r) dr \right) W dx$$

$$W^T \Phi(t) \simeq F^T \Phi(t) + \int_0^t \Phi^T(x) K h I W dx$$

$$W^T \Phi(t) \simeq F^T \Phi(t) + \int_0^t \Phi^T(x) h K W dx$$

$$W^T \Phi(t) \simeq F^T \Phi(t) + W^T (hK)^T \int_0^t \Phi(x) dx$$

$$W^T \Phi(t) \simeq F^T \Phi(t) + W^T (hK)^T Q \Phi(t)$$

$$W^T \simeq F^T + W^T (hK)^T Q$$

$$W^T - W^T (hK)^T Q \simeq F^T$$

$$W^T (I - (hK)^T Q) \simeq F^T$$

So, by setting $N = (I - (hK)^T Q)^T$ and replacing \simeq by $=$, we have

$$NW = F.$$

Which is a linear system of equations with upper triangular coefficients matrix that gives the approximate modified block pulse coefficient of the unknown $X(t)$.

3.2 Error Analysis

In the following theorems, for simplicity we assume $T = 1$ and $h = \frac{1}{m}$.

Theorem 3.2.1 If $\hat{f}_m(t) = \sum_{i=0}^m f_i \psi_i(t)$ and $f_i = \frac{1}{\Delta(I_i)} \int_0^1 f(t) \psi_i(t) dt, i = 0, \dots, m$ then:

- i. $\delta = \int_0^1 (f(t) - \sum_{i=0}^m f_i \psi_i(t))^2 dt$, achieves its minimum value.
- ii. $\{\hat{f}_m(t)\}$ approaches $f(t)$ pointwise.
- iii. $\int_0^1 f^2(t) dt = \sum_{i=0}^\infty f_i^2 \|\psi_i\|^2$

PROOF. Proof is like similar theorem in [3] but intervals of integration have to redefine as $I_i, i = 0, \dots, m$ in (3.1). ■

Theorem 3.2.2 Assume:

- i. $f(t)$ is continuous and differentiable in $[-h, 1 + h]$ with bounded derivative, that is $|f'(t)| < M$.
- ii. $\hat{f}_{\frac{ih}{k}}(t), i = 0, \dots, k-1$, are correspondingly BPFs, $\frac{h}{k}$ MBPFs ..., $\frac{(k-1)h}{k}$ MBPFs expansions of $f(t)$ base on $m+1$ ϵ MBPFs over interval $[0, 1)$.
- iii. $\bar{f}(t) = \frac{1}{k} \sum_{i=0}^{k-1} \hat{f}_{\frac{ih}{k}}(t)$.

Then

$$\left\| f(t) - \hat{f}_{\frac{ih}{k}}(t) \right\| = \mathcal{O}(h), \quad \text{and} \quad \left\| f(t) - \bar{f}(t) \right\| = \mathcal{O}\left(\frac{h}{k}\right) \text{ in } [h, 1 - h].$$

PROOF. Trapezoidal rule for integral is

$$\begin{aligned} \int_a^b f(t) dt &= \frac{b-a}{2} (f(a) + f(b)) - \frac{(b-a)^3 f''(\eta)}{12} \\ &= \frac{b-a}{2} (f(a) + f(b)) + E, \quad \eta \in [a, b], \end{aligned} \quad (4)$$

where E is error of integration. Suppose $t_i = \frac{i}{m} = ih$ and $I_i = [t_{i-1}, t_i]$. The representation error when $f(t)$ is represented by a series of BPFs over every subinterval $[t_i, t_i + \frac{h}{k}]$, $i = 0, \dots, m-1$ is

$$e_i(t) = f(t) - f_i \psi_i(t) = f(t) - f_i,$$

where $f_i = \frac{1}{h} \int_{ih}^{(i+1)h} f(t) dt$. From (4),

$$f_i = \frac{1}{2} (f(t_i) + f(t_i + h)) + E.$$

It is obvious that if $f(t) = C$ (constant), then $e_i(t) = 0$. So, this error is computed for $f(t) = t$ in interval $[t_i, t_i + \frac{h}{k}]$, $i = 1, \dots, m-1$.

For this function $E = 0$, so

$$e_i(t)_{[t_i, t_i + \frac{h}{k}]} = |t - f_i| = \left| t - \frac{t_i + t_{i+1}}{2} \right| = \left| t - \left(t_i + \frac{h}{2} \right) \right| \leq \frac{h}{2}.$$

Then this error with BPFs is $\frac{h}{2} M$.

Similarly, the error when $f(t)$ is represented in a series of ε MBPFs over every subinterval $[t_i, t_i + \frac{h}{k}]$ is

$$\begin{aligned} e_i(t)_{[t_i, t_i + \frac{h}{k}]} &= \left| t - \left(\frac{\sum_{j=0}^{k-1} \left(t_i - \left(\frac{jh}{k} \right) + t_{i+1} - \left(\frac{jh}{k} \right) \right)}{2k} \right) \right| \\ &= \left| t - \left(\frac{\sum_{j=0}^{k-1} \left(t_i - \left(\frac{jh}{k} \right) + t_i + h - \left(\frac{jh}{k} \right) \right)}{2k} \right) \right| \\ &= \left| t - \left(t_i + \frac{h}{2} \right) - \frac{(k-1)h}{2k} \right| \\ &\leq \frac{h}{2k}. \end{aligned}$$

So, the error with ε MBPFs is $\frac{h}{2k} M$.

For I_0 in $[0, \frac{h}{k}]$ we have

$$\begin{aligned} e_i(t)_{[0, \frac{h}{k}]} &= \left| t - \sum_{j=0}^{k-1} \frac{h - \left(\frac{jh}{k} \right)}{2k} \right| \\ &= \left| t - \left(\frac{h}{2} - \frac{(k-1)h}{4k} \right) \right| \\ &= \left| t - \left(\frac{h}{4} + \frac{h}{4k} \right) \right| \\ &= \mathcal{O}\left(\frac{h}{4}\right). \end{aligned}$$

So, the error is $\mathcal{O}\left(\frac{h}{4}\right)$ also for I_n .

Now,

$$\begin{aligned}
 \|e_i(t)\|^2 &= \int_{t_i}^{t_i + \frac{h}{k}} |e_i(t)|^2 dt \\
 &= \int_{t_i}^{t_i + \frac{h}{k}} \frac{h^2}{4k^2} M^2 dt \\
 &= \frac{h^3}{4k^3} M^2, \\
 \|e\|^2 &= \int_0^1 e^2(t) dt \\
 &= \int_0^1 \left(\sum_{i=1}^m \sum_{j=0}^{k-1} e_i(t) \right)^2 dt \\
 &= \sum_{i=1}^m \sum_{j=0}^{k-1} \int_0^1 e_i^2(t) dt \\
 &= \sum_{i=1}^m \sum_{j=0}^{k-1} \|e_i(t)\|^2 \\
 &= \frac{1}{h} \cdot k \cdot \frac{h^3}{4k^3} M^2 \\
 &= \frac{h^2}{4k^2} M^2.
 \end{aligned}$$

We define the representation error between $f(s, t)$ and its 2D- ε MBPFs expansion f_{ij} over every subregion D_{ij} , is defined as

$$e_{ij}(s, t) = f(s, t) - f_{ij},$$

where $D_{ij} := \left\{ (s, t) \mid t_i \leq s \leq t_i + \frac{h}{k}, t_j \leq t \leq t_j + \frac{h}{k} \right\}$.

With Taylor's expansion and similarity to the above discussion,

$$\|e(s, t)\| = \frac{h}{2k} M. \quad \blacksquare$$

Theorem 3.2.3 Assume that

$$P(\omega \in \Omega : \|u(\omega, t)\| < C) = 1.$$

$$\|k_i\| < C, i = 1, 2.$$

Then

$$\begin{aligned}
 \sup_{0 \leq t \leq T} (E(\|(u - \bar{u})\|^2))^{\frac{1}{2}} &= \mathcal{O}\left(\frac{h}{k}\right), \quad t \in [h, 1 - h].
 \end{aligned}$$

PROOF. For a complete proof see [6]. \blacksquare

3.3 Numerical Examples

Example 3.3.1 Consider the following mixed linear Volterra-Fredholm integral equation,

$$X(t) = \cos t - 2t + \int_0^t \int_0^\pi (s-r)X(r)drds,$$

where the exact solution is $f(t) = \cos t$.

Table 1: The Exact and Approximation Solutions of Example 1

t	$m = 1025$	
	Exact	Approximation
0	1	1
0.6	0.8253	0.8224
1.2	0.3626	0.3612
1.8	-0.2276	-0.2272
2.4	-0.7372	-0.5052
3	-0.99	-0.9948

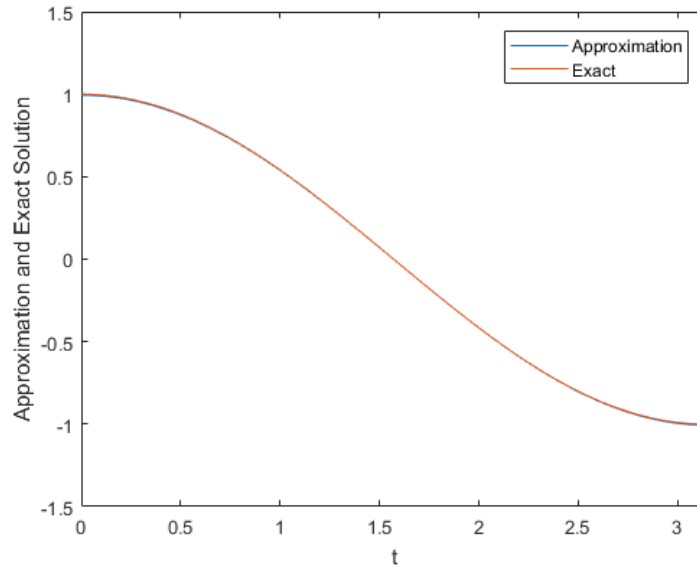


Figure 1: The Trajectory of The Exact and Approximation Solution of Example 1

Based on the graph above, it can be seen that the approximate and exact results coincide with each other at the interval $[0, \pi)$. At the beginning and end that if the number of partitions we take is getting bigger and ε getting smaller, it will produce an approximation result that is closer to the exact.

Example 3.3.2 Consider the following mixed linear Volterra-Fredholm integral equation,

$$X(t) = \cos t + \sin t - t^2 + \frac{\pi}{2}t + \int_0^t \int_0^{\frac{\pi}{2}} (s-r)X(r)drds,$$

where the exact solution is $f(t) = \cos t$.

Table 2: The Exact and Approximation Solutions of Example 2

t	$m = 1025$	
	Exact	Approximation
0	1	1.001
0.3	1.251	1.254
0.6	1.39	1.391
0.9	1.405	1.406
1.2	1.294	1.295
1.5	1.068	1.068

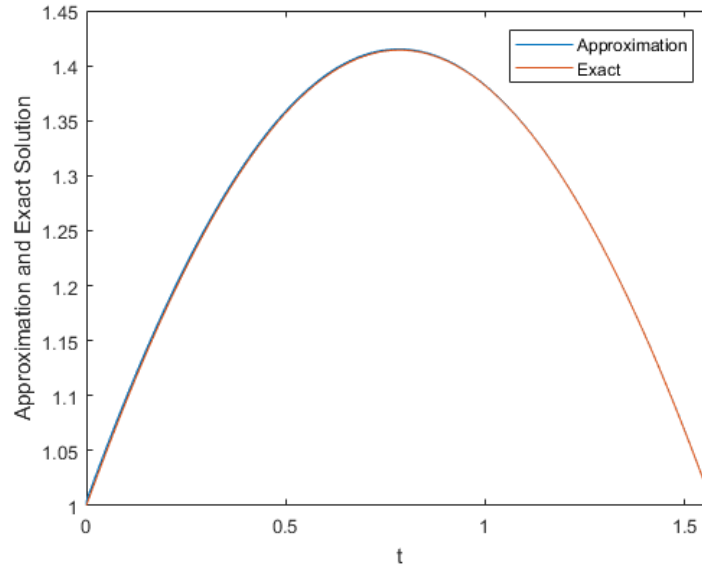


Figure 2: The Trajectory of The Exact and Approximation Solution of Example 2

Based on the graph above, it can be seen that the approximate and exact results coincide with each other at the interval $\left[0, \frac{\pi}{2}\right)$. At the beginning and end that if the number of partitions we take is getting bigger and ε getting smaller, it will produce an approximation result that is closer to the exact.

4 Conclusions

The modified block pulse functions have better results by taking the partition size (m) and $\varepsilon = \frac{1}{a}$ (provided $a > m$ so that $\frac{1}{a} < h$) than block pulse functions. The modified block pulse functions and their integration operational matrix are used to obtain the solution of mixed linear Volterra-Fredholm integral equations. Its applicability and accuracy of the proposed method was checked on some numerical examples. The results show that the value at any point of approximate and exact solutions has very little margin. We also illustrate both results to demonstrate of the proposed method have a good of efficiency and accuracy.

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