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Analytic Method And Matrix Diagonalization On Eigen System Of Hermitian Matrix Operator

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Abstract
The solution of the Hermitian eigenoperator matrix problem produces an eigensystem consisting of eigenvalues and eigenvectors. This study aims to determine the complete solution of the eigensystem and the diagonalization of the Hermitian order matrix operator. $2 \leq n \leq 3$ *analytically. The results of the study show that every eigenproblem in the Hermitian matrix operator* $\mathcal{H}$ *generate several eigenvalues* λ_n $(n = 1, 2, \cdots)$ *according to the order of the matrix operator, the eigenvalues are real numbers. Eigenvectors,* ψ_1, ψ_2, ψ_3 *of the Hermitian matrix operators are orthogonal because* $\lambda_1 \neq \lambda_2 \neq \lambda_3$ *and* $\psi_1^{+}\psi_2 = \psi_1^{+}\psi_3 = \psi_2^{+}\psi_3 = 0$ *thus forming a basis matrix* $= [\psi_1\ \psi_2\ \psi_3]$ *and is unitary. A Hermitian matrix can be diagonalized through its basis matrices and a diagonal matrix is obtained.* $= U^{-1}\mathcal{H}U$ *whose diagonal elements are the eigenvalues of the Hermitian operator.*
Keywords: *Matrix; eigensystem; Hermitian matrix*

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INTRODUCTION

Physics is a natural science that studies the elements that form the universe and the forces that act within it, from macroscopic to microscopic scales. [1]. In general, physics can be divided into 2 branches: Classical Physics and Quantum Physics. Classical Physics began in 1800-1890 with the development of classical mechanics, heat physics, electrodynamics, classical thermodynamics, magnetism, electricity, and waves. [2]. Quantum Physics studies the behavior of matter and energy at the molecular, atomic, and nuclear levels, or at the microscopic level. [3]. The study of quantum physics is so abstract that it requires the postulates of quantum mechanics.

Some of the postulates that underlie the formulation of quantum physics include state representation, representation of dynamic variables, system evolution, and motion constants. [4].

Quantum physics has continued to evolve since Erwin Schrödinger formulated the Schrödinger equation in 1925 [5]. One of the mathematical formulations used in quantum physics, besides the Schrödinger equation, is matrix mechanics [6]. This formulation is used to determine the transmission coefficients for three potential barriers in GaN, SiC, and GaAs [7]. Analysis of transmission coefficients can also be performed using the matrix propagation method for a three-potential-barrier structure composed of GaAs, GaSb, and AlAs semiconductors, which exhibit certain variations [8]. Nasiroh et al. (2020) also used the same method to determine the transmission coefficients for four potential barriers [9]. The matrix propagation method can also be used to determine the probability of electrons penetrating potential barriers in superlattice structures, whose magnitudes depend on the transmission coefficients [10].

Matrix mechanics is a fundamental theory with many applications, not only in mathematics and physics but also in engineering, computer science, and technology [11]. Matrices are used in various fields to simplify complex problems [12]. In computer science and technology, one of its applications is used to send secret messages [13]. An example is the Hill Cipher, a block cipher algorithm that uses an $n \times n$ matrix as a key, where n represents the block size [14]. This application demonstrates that matrix mechanics theory can support the development of information system security. Additionally, various matrix-based algorithms have been developed to meet the needs of modern computing.

One of the problems with using matrix operators is the eigenvalue problem. The eigenproblem arises when an operator is imposed on a function. [15], Then the function does not change its condition but is multiplied by certain constants [17] The eigen problem of a matrix operator is a characteristic problem of a matrix operator, because the eigenvalue of a matrix operator is the distinctive value of a rectangular matrix of order n [18]. The order of the matrix operator in the eigenproblem determines the difficulty of solving the eigenproblem analytically. [19]. The greater the order, the more difficult it is to determine the matrix's error value, so the right formula is needed. [20]. Mathematically, the eigen problem is formulated as follows:

$$\hat{A}\psi = \lambda\psi \tag{1}$$

with $\hat{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ is the eigen operator

$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}$ is the vector or eigenfunction of the operator A in the form of a nonzero column matrix

λ is the eigenvalue of the operator A [21]

The eigenvalues of operators in quantum physics correspond to observable quantities, so the eigenvalues must be real numbers. [22]. The eigen problem that produces real number eigenvalues is the eigen problem of a Hermitian operator. [23]. A Hermitian operator is a linear operator in the same Hilbert space as its own adjoint. [24] which means that the Hermitian operator satisfies the following equation:

$$\left(\hat{A}\,\varphi, \psi\right) = \left(\varphi, \hat{A}^{+}\psi\right) \tag{2}$$

or $A = A^{+}$. The above equation ensures that the eigenvalues of the Hermitian operator are real numbers, so the eigenfunctions (vectors) of the Hermitian operator are orthogonal. [25]. The orthogonality of the eigenfunctions associated with an eigenvalue of the Hermitian operator implies that if a system is measured in one state, it cannot be in another state corresponding to a different eigenvalue simultaneously. [26].

Hemitian matrices, $\mathcal{H}_{nxn}$ They are fundamental to quantum mechanics because they describe operators with real eigenvalues. The Hemitian matrix is a complex square matrix. [27] This matrix satisfies:

$$\mathcal{H}^{+} = \mathcal{H} \text{ atau } (\mathcal{H}^{*})^{T} = \mathcal{H}. \tag{3}$$

According to the spectral theorem, the Hemitian matrix can be diagonalized by a unitary matrix, U of the same degree through equation:

$$D = U^{-1}\mathcal{H}U \tag{4}$$

Where D is a diagonal matrix of degree n with diagonal elements of real numbers, U is a unitary matrix and U^{-1} Is the inverse of the unitary matrix [28]. In addition, Hermitian matrices have mutually orthogonal eigenvectors corresponding to distinct eigenvalues. [29]. Even if there are degenerate eigenvalues, it is possible to find an orthogonal basis C^{n} Which consists of n eigenvectors $\mathcal{H}$.

The eigenproblem by the Hermitian matrix operator, $\mathcal{H}\psi = \lambda\psi$ Can be expressed in the form of its characteristic equation, i.e. :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} = \lambda \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}$$

or

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5}$$

The matrix operator $\mathcal{H}$ Which has an n-degree has n eigenvalues and eigenfunctions. The eigenvalues and eigenfunctions of a Hermitian matrix operator are $\mathcal{H}$ They are obtained by solving the characteristic equation of the operator [30]. The eigenfunctions of Hermitian operators are orthogonal so that they can form the basis matrix of the operator in question [31].

Several researchers have conducted research on eigenproblems for matrix operators. For example, Pena (2025) developed a mathematical method for estimating the eigenvalues of positive symmetric Toeplitz matrices [32]. Furthermore, Deng (2024) found exact solutions for finding eigenvalues and eigenvectors for several-dimensional matrices, especially those related to one-dimensional Laplacian operators. Research on the eigenvalue problem for Hermitian matrices remains relatively limited, especially in an analytical setting. Therefore, this study aims to solve the eigenvalue problem of Hermitian matrix operators of order using analytical methods and matrix diagonalization.

METHOD

This research is a non-experimental study. The method used is a literature review to apply existing theories to solve the eigenproblem of the Hermitian matrix operator analytically and to diagonalize the matrix. The research method consists of several steps as follows:

1. Testing matrix operators $\mathcal{H}$ of degrees 2 and 3 is a Hermitian matrix, $\mathcal{H}^{+} = \mathcal{H}$

2. Determine the eigenvalues of the Hermitian matrix operator by solving the eigencharacteristic equation,

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ atau } (\mathcal{H}-\lambda I)\psi = 0$$

by setting the value or determinant of $(\mathcal{H}-\lambda I)$ equal to zero or $|(\mathcal{H}-\lambda I)| = 0$.

3. Construct unnormalized eigen functions by subsuming the corresponding eigenvalues into the eigen characteristic equation. For a matrix operator of order 2, there are two eigenfunctions, namely:

$$\psi_1 = \psi \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \psi_2 = \psi \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

While for the matrix operator of order three there are three eigenfunctions, namely :

$$\psi_1 = \psi \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix};\ \psi_2 = \psi \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ and } \psi_3 = \psi \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

4. Determine the normalization constant of the eigenfunctions by applying the normalization condition,

$$\psi^{+}\psi = 1$$

5. Construct a modal matrix from the eigenfunctions of a Hermitian matrix operator,

$$U = [\psi_1 \quad \psi_2] \text{ or } U = \psi \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \text{ for matrix operators of degree 2}$$

and $U = [\psi_1 \quad \psi_2 \quad \psi_3]$ or $= \psi \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ for matrix operators of degree 3

6. Testing the basis matrix U is a unitary matrix, $U^{-1} = U^{+}$
7. Determine the diagonal matrix of the Hermitian matrix operator by using the equation

$$D = U^{-1}\mathcal{H}U$$

RESULTS AND DISCUSSION

The research results on solving the eigen system of hermitian matrix operators using analytic methods and matrix diagonalization, including eigenvalues and eigenfunctions (eigenvalues). Presented hermitian matrix operator of order 2 and 3, which is as follows:

A. Eigen problems by matrix operators $\mathcal{H} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

The solution to the eigen problem of the Hermitian operator of a matrix of degree 2 is :

(i) A matrix $\mathcal{H} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is a Hermitian matrix if $\mathcal{H}^{+} = \mathcal{H}$.

Evidence $\mathcal{H}^{+} = (\mathcal{H}^{*})^{T} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^{T} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

(ii) Determine the eigenvalues of its eigen characteristic equation,

$\begin{bmatrix} -\lambda & i \\ -i & -\lambda \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This equation will have a solution if $\begin{vmatrix} -\lambda & i \\ -i & -\lambda \end{vmatrix} = 0$ or

$\lambda^2 - 1 = 0$ then $\lambda_{1,2} = \sqrt{1}$. So the eigenvalue of the matrix operator $\mathcal{H}$ is $\lambda_1 = 1$ and $\lambda_2 = -1$

(iii) Substitute the eigenvalue $\lambda_1 = 1$ To the characterization equation obtained:

$\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $-\psi_1 + i\psi_2 = 0$. If $\psi_2 = t$ then $\psi_1 = i$

So the unnormalized eigen vector is $\Psi_1 = t\begin{bmatrix} i \\ 1 \end{bmatrix}$. By applying the normalization $\psi^{+}\psi = 1$, we obtain the normalized eigen vector of the Hermitian operator $\mathcal{H} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ for the eigenvalue $\lambda = 1$ adalah $\Psi_1 = \sqrt{\frac{1}{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$.

As for the eigenvalue $\lambda_2 = -1$, the normalized eigen vector is $\Psi_2 = \sqrt{\frac{1}{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

(iv) Since $\lambda_1 \neq \lambda_2$ and $\Psi_1^{+}\,\Psi_2 = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \end{bmatrix}\right\}\frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix} = 0$ The eigen vactors of the matrix operator are orthogonal. Hence, the eigen vectors form the modal matrix $U = [\Psi_1 \quad \Psi_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

(v) Determine the inverse and adjoint of the modal matrix $U = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$. The inverse of the matrix is formulated: $U^{-1} = \frac{1}{\det U}\, adj\, U$. Thus obtained $U^{-1} = \frac{1}{i}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$ And the adjoint of the capital matrix is $U^{+} = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\right\}^{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$.

(vi) Since $U^{-1} = U^{+}$ then $U = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ Is a unitary matrix. Then the diagonalization of the Hermitian matrix operator is

$$D = U^{-1}\mathcal{H}U = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

B. Eigenproblems by matrix operators $\mathcal{H} = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$

The solution to the eigen problem of the Hermitian operator of a matrix of degree 2 is :

(i) A matrix $\mathcal{H} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is a Hermitian matrix if $\mathcal{H}^{+} = \mathcal{H}$.

Evidence $\mathcal{H}^{+} = (\mathcal{H}^{*})^{T} = \begin{bmatrix} 1 & 1-i \\ 1+i & 1 \end{bmatrix}^{T} = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$

(ii) Determine the eigenvalues of its eigen characteristic equation,

$\begin{bmatrix} 1-\lambda & 1+i \\ 1-i & 1-\lambda \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This equation will have a solution if $\begin{vmatrix} 1-\lambda & 1+i \\ 1-i & 1-\lambda \end{vmatrix} = 0$ or

$\lambda^2 - 2\lambda - 1 = 0$ then $\lambda_{1,2} = 1 \pm \sqrt{2}$. So the eigenvalue of the matrix operator $\mathcal{H} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is $\lambda_1 = 1 + \sqrt{2}$ dan $\lambda_2 = 1 - \sqrt{2}$

(iii) Substitute the eigenvalue $\lambda_1 = 1 + \sqrt{2}$ To the characterization equation obtained :

$\begin{bmatrix} -\sqrt{2} & 1+i \\ 1-i & -\sqrt{2} \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $-\psi_1\sqrt{2} + \psi_2(1+i) = 0$. If $\psi_2 = t$ then $\psi_1 = \frac{t}{\sqrt{2}}(1+i)$

So the unnormalized eigen vector is $\Psi_1 = t\begin{bmatrix} \frac{1}{\sqrt{2}}(1+i) \\ 1 \end{bmatrix}$. By applying the normalization condition $\psi^{+}\psi = 1$, we obtain the normalized eigen vector of the Hermitian operator $\mathcal{H} = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$ for the eigenvalue $\lambda = 1+\sqrt{2}$ is $\Psi_1 = \sqrt{\frac{1}{2}}\begin{bmatrix} \frac{1}{\sqrt{2}}(1+i) \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i \\ \sqrt{2} \end{bmatrix}$.

As for the eigenvalue $\lambda_2 = 1-\sqrt{2}$, the normalized eigen vector is

$$\Psi_2 = \sqrt{\frac{1}{2}}\begin{bmatrix} -\frac{1}{\sqrt{2}}(1+i) \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1-i \\ \sqrt{2} \end{bmatrix}$$

(iv) Since $\lambda_1 \neq \lambda_2$ and $\Psi_1^{+}\,\Psi_2 = \left\{\frac{1}{2}\begin{bmatrix} 1-i & \sqrt{2} \end{bmatrix}\right\}\frac{1}{2}\begin{bmatrix} -1-i \\ \sqrt{2} \end{bmatrix} = 0$ The eigen vectors of the matrix operator are orthogonal. Hence, the eigen vectors form the modal matrix. $U = \begin{bmatrix} \Psi_1 & \Psi_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i & -1-i \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

(v) Determine the inverse and adjoint of the modal matrix. $U = \frac{1}{2}\begin{bmatrix} 1+i & -1-i \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$. The 2nd-order matrix inverse can be formulated: $U^{-1} = \frac{1}{\det U} adj\ U$. Thus obtained

$$U^{-1} = \frac{2}{\sqrt{2}(1+i)}\frac{1}{2}\begin{bmatrix} \sqrt{2} & 1+i \\ -\sqrt{2} & 1+i \end{bmatrix} = \frac{1}{1+i}\begin{bmatrix} 1 & \frac{1+i}{\sqrt{2}} \\ -1 & \frac{1+i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+i} & \frac{1}{\sqrt{2}} \\ -\frac{1}{1+i} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-i & \sqrt{2} \\ -1+i & \sqrt{2} \end{bmatrix}$$

And the adjoint capital matrix is $U^{+} = \left\{\frac{1}{2}\begin{bmatrix} 1-i & -1+i \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\right\}^{T} = \frac{1}{2}\begin{bmatrix} 1-i & \sqrt{2} \\ -1+i & \sqrt{2} \end{bmatrix}$.

(vi) Since $U^{-1} = U^{+}$ then $U = \frac{1}{2}\begin{bmatrix} 1+i & -1-i \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ Is a unitary matrix. Then the diagonalization of the Hermitian matrix operator is

$$D = U^{-1}\mathcal{H}U = \frac{1}{2}\begin{bmatrix} 1-i & \sqrt{2} \\ -1+i & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1+i & -1-i \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1-i & \sqrt{2} \\ -1+i & \sqrt{2} \end{bmatrix}\frac{1}{2}\begin{bmatrix} (1+\sqrt{2})(1+i) & (\sqrt{2}-1)-i(\sqrt{2}-1) \\ 2+\sqrt{2} & -2+\sqrt{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4+4\sqrt{2} & 0 \\ 0 & 4-4\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$$

C. Eigenproblems by matrix operators $\mathcal{H} = \begin{bmatrix} 1 & i & 1 \\ -i & 1 & -2i \\ 1 & 2i & 1 \end{bmatrix}$

The solution to the eigen problem of the Hermitian operator of a matrix of degree 2 is :

(i) A matrix $\mathcal{H} = \begin{bmatrix} 1 & i & 1 \\ -i & 1 & -2i \\ 1 & 2i & 1 \end{bmatrix}$ is a Hermitian matrix if $\mathcal{H}^{+} = \mathcal{H}$.

Evidence $\mathcal{H}^{+} = (\mathcal{H}^{*})^{T} = \begin{bmatrix} 1 & -i & 1 \\ i & 1 & 2i \\ 1 & -2i & 1 \end{bmatrix}^{T} = \begin{bmatrix} 1 & i & 1 \\ -i & 1 & -2i \\ 1 & 2i & 1 \end{bmatrix}$

(ii) Determine the eigenvalues of its eigen characteristic equation,

$\begin{bmatrix} 1-\lambda & i & 1 \\ -i & 1-\lambda & -2i \\ 1 & 2i & 1-\lambda \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$. This equation will have a solution if

$$\begin{vmatrix} 1-\lambda & i & 1 \\ -i & 1-\lambda & -2i \\ 1 & 2i & 1-\lambda \end{vmatrix} = 0 \text{ or } -\lambda^3 + 3\lambda^2 + 3\lambda - 1 = 0$$

The solution to the above equation is : $\lambda_1 = 2+\sqrt{3}$, $\lambda_2 = -1$. *and* $\lambda_3 = 2-\sqrt{3}$. So the eigenvalue of the matrix operator $\mathcal{H}$ is $\lambda_1 = 2+\sqrt{3}$; $\lambda_2 = -1$ dan $\lambda_3 = 2-\sqrt{3}$.

(iii) Determine the eigen vector for each eigen value by substituting it into the characteristic equation.

a. $\lambda_1 = 2+\sqrt{3}$ Obtained: $\begin{bmatrix} -1-\sqrt{3} & i & 1 \\ -i & -1-\sqrt{3} & -2i \\ 1 & 2i & -1-\sqrt{3} \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\left(-1-\sqrt{3}\right)\psi_1 + i\psi_2 + \psi_3 = 0 \quad (6)$$

$$-i\psi_1 - \left(1+\sqrt{3}\right)\psi_2 - 2i\psi_3 = 0 \quad (7)$$

$$\psi_1 + 2i\psi_2 - \left(1+\sqrt{3}\right)\psi_3 = 0 \quad (8)$$

By the method of substitution and elimination or using Cramer's rule, the solution of the System of Linear Equations (SPL) (6), (7), and (8) above is: $\psi_1 = t\left(-1+\sqrt{3}\right)$; $\psi_2 = -it$; and $\psi_3 = t$. So the unnormalized eigen vector of is $\Psi_1 = t\begin{bmatrix}-1+\sqrt{3}\\ -i\\ 1\end{bmatrix}$. By applying the normalization condition $\psi^{+}\psi = 1$, we obtain the normalized eigen vector of the Hermitian operator $\mathcal{H}$ for the eigenvalue $\lambda = 2+\sqrt{3}$ is $\Psi_1 = \sqrt{\frac{1}{6-2\sqrt{3}}}\begin{bmatrix}-1+\sqrt{3}\\ -1\\ 1\end{bmatrix}$.

b. $\lambda_2 = -1$ Obtained: $\begin{bmatrix} 2 & i & 1 \\ -i & 2 & -2i \\ 1 & 2i & 2 \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$2\psi_1 + i\psi_2 + \psi_3 = 0 \quad (9)$$

$$-i\psi_1 + 2\psi_2 - 2i\psi_3 = 0 \quad (10)$$

$$\psi_1 + 2i\psi_2 + 2\psi_3 = 0 \quad (11)$$

he solutions of SPL (9), (10), and (11) give : psi sub 1 equals 0 , psi sub 2 equals i, next row, i, next row, 1 end matrix close bracketd $\psi_3 = t$. So the unnormalized eigen vector is $\Psi_2 =$

$t\begin{bmatrix}0\\i\\1\end{bmatrix}$. By applying the normalization condition $\psi^{+}\psi = 1$, we obtain the normalized eigen vector of the Hermitian operator $\mathcal{H}$ for the eigen value $\lambda = 2+\sqrt{3}$ is $\Psi_2 = \sqrt{\frac{1}{2}}\begin{bmatrix}0\\i\\1\end{bmatrix}$

c. $\lambda_3 = 2-\sqrt{3}$ obtained : $\begin{bmatrix}-1+\sqrt{3} & i & 1\\ -i & -1+\sqrt{3} & -2i\\ 1 & 2i & -1+\sqrt{3}\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\left(-1+\sqrt{3}\right)\psi_1 + i\psi_2 + \psi_3 = 0 \tag{12}$$

$$-i\psi_1 - \left(1-\sqrt{3}\right)\psi_2 - 2i\psi_3 = 0 \tag{13}$$

$$\psi_1 + 2i\psi_2 - \left(1-\sqrt{3}\right)\psi_3 = 0 \tag{14}$$

The solutions of SPL (12), (13) and (14) above are : $\psi_1 = t\left(-1-\sqrt{3}\right)$; $\psi_2 = -it$; and $\psi_3 = t$. So the unnormalized eigen vector is $\Psi_3 = t\begin{bmatrix}-1-\sqrt{3}\\ -i\\ 1\end{bmatrix}$. By applying the normalization condition $\psi^{+}\psi = 1$, we obtain the normalized eigen vector of the Hermitian operator $\mathcal{H}$ for thr eigen value $\lambda = 2-\sqrt{3}$ adalah $\Psi_3 = \sqrt{\frac{1}{2(3+\sqrt{3})}}\begin{bmatrix}-1-\sqrt{3}\\ -i\\ 1\end{bmatrix}$

(iv) Since (a) $\lambda_1 \neq \lambda_2$ and $\Psi_1^{+}\,\Psi_2 = \left\{\frac{1}{\sqrt{6-2\sqrt{3}}}\begin{bmatrix}-1+\sqrt{3} & i & 1\end{bmatrix}\right\}\frac{1}{\sqrt{2}}\begin{bmatrix}0\\i\\1\end{bmatrix} = 0$

(b) $\lambda_1 \neq \lambda_3$ and $\Psi_1^{+}\,\Psi_3 = \left\{\frac{1}{\sqrt{6-2\sqrt{3}}}\begin{bmatrix}-1+\sqrt{3} & i & 1\end{bmatrix}\right\}\sqrt{\frac{1}{2(3+\sqrt{3})}}\begin{bmatrix}-1-\sqrt{3}\\ -i\\ 1\end{bmatrix} = 0$

(c) $\lambda_2 \neq \lambda_3$ and $\Psi_2^{+}\,\Psi_3 = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i & 1\end{bmatrix}\right\}\sqrt{\frac{1}{2(3+\sqrt{3})}}\begin{bmatrix}-1-\sqrt{3}\\ -i\\ 1\end{bmatrix} = 0$

Then the eigenvectors of the matrix operator are orthogonal. Then the eigen vectors form a modal matrix $U = [\psi_1 \quad \psi_2 \quad \psi_3] = \begin{bmatrix}\frac{-1+\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & 0 & -\frac{1+\sqrt{3}}{\sqrt{6+2\sqrt{3}}}\\ -\frac{i}{\sqrt{6-2\sqrt{3}}} & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{6+2\sqrt{3}}}\\ \frac{1}{\sqrt{6-2\sqrt{3}}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6+2\sqrt{3}}}\end{bmatrix}$

(v) Since $U^{-1} = U^{+} = \begin{bmatrix}\frac{-1+\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & \frac{i}{\sqrt{6-2\sqrt{3}}} & \frac{1}{\sqrt{6-2\sqrt{3}}}\\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1+\sqrt{3}}{\sqrt{6+2\sqrt{3}}} & \frac{i}{\sqrt{6+2\sqrt{3}}} & \frac{1}{\sqrt{6-2\sqrt{3}}}\end{bmatrix}$ then U is a unitary matrix. Then the diagonalization of the Hermitian matrix operator is

$$D = U^{-1}\mathcal{H}U = \begin{bmatrix} \frac{-1+\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & \frac{i}{\sqrt{6-2\sqrt{3}}} & \frac{1}{\sqrt{6-2\sqrt{3}}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1+\sqrt{3}}{\sqrt{6+2\sqrt{3}}} & \frac{i}{\sqrt{6+2\sqrt{3}}} & \frac{1}{\sqrt{6-2\sqrt{3}}} \end{bmatrix} \begin{bmatrix} 1 & i & 1 \\ -i & 1 & -2i \\ 1 & 2i & 1 \end{bmatrix} \begin{bmatrix} \frac{-1+\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & 0 & -\frac{1+\sqrt{3}}{\sqrt{6+2\sqrt{3}}} \\ -\frac{i}{\sqrt{6-2\sqrt{3}}} & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{6+2\sqrt{3}}} \\ \frac{1}{\sqrt{6-2\sqrt{3}}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6+2\sqrt{3}}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1+\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & \frac{i}{\sqrt{6-2\sqrt{3}}} & \frac{1}{\sqrt{6-2\sqrt{3}}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1+\sqrt{3}}{\sqrt{6+2\sqrt{3}}} & \frac{i}{\sqrt{6+2\sqrt{3}}} & \frac{1}{\sqrt{6-2\sqrt{3}}} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & 0 & \frac{1-\sqrt{3}}{\sqrt{6+2\sqrt{3}}} \\ -i\frac{2+\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & -\frac{i}{\sqrt{2}} & -i\frac{2-\sqrt{3}}{\sqrt{6+2\sqrt{3}}} \\ \frac{2+\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & -\frac{1}{\sqrt{2}} & \frac{2-\sqrt{3}}{\sqrt{6+2\sqrt{3}}} \end{bmatrix}$$

$$= \begin{bmatrix} 2+\sqrt{3} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$$

Matrix operators in quantum mechanics are quite varied in type, but not all of them are Hermitian. A Hermitian matrix operator is a rectilinear matrix operator with real numbers on its main diagonal and other elements that can be real and/or imaginary (complex numbers) and satisfies the requirement $\mathcal{H}^+ = \mathcal{H}$. According to Jurić's study (2022), the Hermitian condition can be reduced to a symmetry condition, guaranteeing that the expectation value of an observable is real [34]. This finding is corroborated by S. Epiningtiyas et al. (2020), who demonstrate that Hermitian operators possess a distinct property: they generate real eigenvalues for all element types, with corresponding eigenfunctions that may be real or complex, contingent upon the element type [15]. For matrices whose elements are real numbers and satisfy the condition $A^+ = A$ Then it is called a symmetric matrix along its diagonal. Therefore, using the matrix function A allows determination of trace bounds that depend solely on the diagonal elements and matrix powers [35]. For eigenvalue problems, both symmetric and Hermitian matrix operators yield real eigenvalues, with the maximum number of eigenvalues generated corresponding to the order of the matrix operator. [15]. Consider a Hermitian matrix operator of degree $n = 2$ the eigenvalues produced are λ_1 and λ_2 with this eigenvector ψ_1 and ψ_2. Similarly, a Hermitian matrix operator of this degree $n = 3$ We obtain this eigenvalue $(\lambda_1, \lambda_2, \lambda_3)$ *and* eigenvector(ψ_1, ψ_2, ψ_3).

Eigenvectors (ψ_1, ψ_2) of a second-order Hermitian matrix operator is orthonal if the resulting eigenvalues are $\lambda_1 \neq \lambda_2$ and psi sub 2, psi sub 3 , close bracket and unitary in nature, cap U to the , cap D equal cap Uo the f thet wos ulting eigton onone are $\lambda_1 \neq \lambda_2 \neq \lambda_3$ dan $\psi_1^+\psi_2 = \psi_1^+\psi_3 = \psi_2^+\psi_3 = 0$. The eigenvectors of the orttoogonatzeroo Herononeeitian matrix operator onean form a basis matrix. $U = [\psi_1, \psi_2, \psi_3]$ and unitary in nature, $U^{-1} = U^+$ [31]. Diagonalization of a Hermitian matrix can be done through the properties of its basis matrices namely $D = U^{-1}\mathcal{H}U$ [28]. Where $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ is a diagonal matrix whose diagonal elements are the eigenvalues of the Hermitian matrix operator. This is supported by research by

Chadzitaskos et al. (2020), which states that the eigenvectors of a Hermitian operator can form an orthonormal basis in a function space, in L^2 [36].

In quantum mechanics, the eigenvalues of Hermitian operators correspond to the values of physical observables, such as energy. $\hat{H}\,\psi = \lambda\psi$, total angular momentum $\hat{L}^2\psi = \lambda\psi$ and the angular momentum component along the $-z$, $\hat{L}_z\psi = \lambda\psi$ [37]. In physics, measurements produce real numbers, and the eigenvalues of Hermitian operators are real. Therefore, the eigenvalues of an operator correspond to the measurement results of an observable. This notion is supported by Surjan et al. (2024), who emphasize the fundamental role of Hermitian operators in quantum mechanics due to their real eigenvalues [38]. Similarly, Juric (2022) highlights that Hermitian matrices possess real eigenvalues and orthogonal eigenvectors, making them ideal for representing physical observables [34].

CONCLUSION

A Hermitian matrix is a square complex matrix whose main diagonal elements are real numbers and has the property. $\mathcal{H}^+ = \mathcal{H}$. The Hermitian matrix operator is one of the operators that can describe the results of observable measurements because it produces real eigenvalues. The sum of the eigenvalues and eigenvectors produced by the Hermitian matrix operator is of order. $2 \leq n \leq 3$ According to the matrix's order. Eigenvectors $(\psi_{1,}\,\psi_2, \psi_3)$ from Hermitian matrix operators $\mathcal{H}$ orthogonal $(\lambda_1 \neq \lambda_2 \neq \lambda_3$ and $\psi_1^+\psi_2 = \psi_1^+\psi_3 = \psi_2^+\psi_3 = 0)$ thus forming a basis matrix $U = [\psi_1 \quad \psi_2 \quad \psi_2]$ and unitary in nature, $U^+ = U^{-1}$. Using its basis matrices, the Hermitian matrix operator can be diagonalized through the equation. $D = U^{-1}\mathcal{H}U$. And D is a diagonal matrix of the same order as the Hermitian matrix operator, and its diagonal elements are the eigenvalues of the Hermitian operator. This finding confirms the fundamental properties of the Hermitian operator in quantum mechanics, especially in the simple case of low-order matrices, and also provides an overview of the analytical process in determining eigenvalues, eigenvectors, eigenvector normalization, and Hermitian matrix diagonalization.

AUTHOR CONTRIBUTIONS

Bambang Supriadi: Conceptualization, Methodology, Validation, and Finalization; Sisilia Nur Hikmah Anggraeni, Badriyah, Fidia Alhikmah Putri, Puput Aprilia Eka Sari: Drafting the introduction, results, and discussion; Indah Selviandri and May Yani Br Sembiring: Theoretical Analysis and Translation.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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