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Comparison Structure Preserving Integrator with Conventional Method in Energy Conservation From Double Pendulum

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Abstract

This study investigates the long-time numerical behavior of a planar double pendulum formulated as a Hamiltonian system under different time integration schemes. The double pendulum is a benchmark chaotic system in which accurate long term energy behavior is essential for obtaining physically meaningful trajectories. The purpose of this work is to compare a conventional explicit method (classical Runge–Kutta), a Störmer–Verlet, and two structure preserving integrators (implicit midpoint and Gauss–Legendre collocation) in terms of phase space dynamics, energy conservation, and computational cost. The equations of motion are integrated numerically with identical physical parameters, time-step size, and simulation horizon for all methods. Phase plots of the angles and canonical momenta are compared with reference results from the literature to verify the correctness of the implementation. Long time energy behavior is then assessed using scalar error metrics derived from the numerical Hamiltonian, and simple runtime benchmarks are used to evaluate efficiency. The results show that the implicit midpoint and the Gauss–Legendre method provide phase portraits consistent with the reference and maintain the energy close to a single level over long times, whereas the classical Runge–Kutta scheme exhibits a mild but systematic drift. The Gauss–Legendre integrator also achieves the smallest energy errors and the highest throughput, making it the most reliable and efficient option among the methods tested. In other words, for long time simulations of Hamiltonian systems, methods that preserve the symplectic structure are more important than merely increasing the formal order of accuracy. These findings provide a quantitative benchmark for geometric integrators on chaotic systems and form a conceptual bridge toward future structure preserving, data driven models such as Hamiltonian Neural Networks.

Keywords: ***double pendulum; Hamiltonian system; symplectic integrator; Gauss–Legendre method; energy conservation***

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INTRODUCTION

Numerical methods are essential for solving complex physics problems when analytical solutions are unattainable [1]. Conventional numerical methods commonly used in simulations include the Euler method and the Fourth-Order Runge-Kutta (RK4). Generally, they fail to preserve the

energy of Hamiltonian systems. Consequently, numerical simulations using these methods often show cumulative energy drift over time [2]. In contrast, symplectic integrators are designed to preserve the Hamiltonian structure. While the total energy may fluctuate slightly at each step, it remains bounded and stable over the long term [3], unlike the continuous drift observed in conventional methods. The superiority of symplectic integrators, such as Störmer–Verlet and Gauss–Legendre, over conventional methods for nonlinear Hamiltonian systems has been well documented (4, 5, & 6).

Despite these known advantages, current numerical studies on the double pendulum predominantly rely on classical RK4 to analyze chaotic behavior, offering only limited quantitative evaluation of long-term energy conservation [6]. Furthermore, existing comparative studies often evaluate a narrow selection of integrators or time steps and rarely quantify the critical trade-off between long-term energy drift and computational cost for the non-separable double pendulum Hamiltonian [7].

To address this gap, this study presents a comprehensive quantitative benchmark comparing structure-preserving integrators–Störmer–Verlet, implicit midpoint, and Gauss–Legendre–against conventional RK methods. The novelty of this work lies in providing a unified framework to systematically evaluate the trade-off between energy conservation and computational efficiency over long integration times, thereby establishing a robust reference for selecting appropriate integrators in complex Hamiltonian simulations.

METHOD

Hamiltonian Mechanics

In this paper, we propose a numerical simulation method based on a coordinate representation of Hamiltonian System. The Hamiltonian function represents the total energy of the system. [6]. In Hamiltonian mechanics, using coordinates $q = (q_1, q_2, \dots, q_n)$ to represent positions of objects and $p = (p_1, p_2, \dots, p_n)$ denotes their momentum [8]. By using the derivative of the Hamiltonian function, the Hamilton equation for a one-dimensional [9] system is expressed as

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \; dan \; \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \tag{1}$$

In this context, the generalized coordinates q correspond to the angular variables of the pendulum, i.e., $q = (\theta_1, \theta_2)$, with corresponding conjugate momenta $p = (p_1, p_2)$. The Hamiltonian system can be written explicitly as a first-order dynamical system:

$$\frac{d}{dt}\begin{pmatrix}\theta_1\\ \theta_2\\ p_1\\ p_2\end{pmatrix} = f(\theta_1, \theta_2, p_1, p_2) \tag{2}$$

where f defines the nonlinear vector field of the system. For numerical implementation, this system is further expressed in explicit form as:

$$\frac{d\theta_i}{dt} = f_i(\theta_1, \theta_2, p_1, p_2), \frac{dp_i}{dt} = f_i(\theta_1, \theta_2, p_1, p_2), i = 1,2 \tag{3}$$

Double Pendulum

The double pendulum consists of two point masses (m_1, m_2) connected by two rigid rods of lengths l_1 and l_2. The system's configuration is fully defined by the two angular displacements of the rods, resulting in two degrees of freedom [10]. In the Hamiltonian framework, the state of the

double pendulum is fully described by four canonical coordinates $(\theta_1, \theta_2, p_1, p_2)$. The Hamiltonian function for this system is defined as:

$$\mathcal{H}(\theta_1, \theta_2, p_1, p_2) = \frac{1}{2}[Ap_1^2 + 2Bp_1p_2 + Cp_2^2] - (m_1 + m_2)gl_1 \cos\theta_1 - m_2gl_2 \cos\theta_2 \quad (4)$$

Define $A = \frac{m_2 l_2^2}{\det M}, B = \frac{-m_2 l_1 l_2 \cos(\theta_1 - \theta_2)}{\det M}, C = \frac{(m_1 + m_2) l_1^2}{\det M}$, and $\det M = m_2 l_1^2 l_2^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))$.

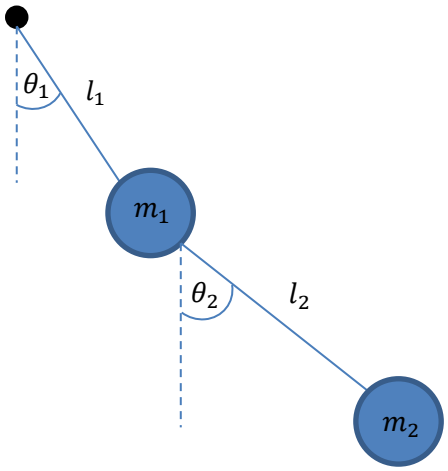


Figure 1. Double Pendulum System

By applying Hamilton's equations, we obtain the angular rates $(\dot{\theta}_1, \dot{\theta}_2)$ and the canonical momentum rates $(\dot{p}_1, \dot{p}_2)$ as functions of the current state. These resulting first-order ordinary differential equations (ODEs) are evaluated repeatedly at each integration step. The explicit formulations below are designed to be directly implemented by integration schemes (both symplectic and conventional) in the numerical experiments:

$$\dot{\theta}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = \frac{l_2 p_1 - l_1 p_2 \cos(\theta_1 - \theta_2)}{l_1^2 l_2 m_1 + l_1^2 l_2 m_2 \sin^2(\theta_1 - \theta_2)} \quad (5)$$

$$\dot{\theta}_2 = \frac{\partial \mathcal{H}}{\partial p_2} = \frac{l_1 m_1 p_2 + l_1 m_2 p_2 - l_2 m_2 p_1 \cos(\theta_1 - \theta_2)}{l_1 l_2^2 m_2^2 \sin^2(\theta_1 - \theta_2) + l_1 l_2^2 m_1 m_2} \quad (6)$$

$$\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial \theta_1} = -(m_1 + m_2)gl_1 \sin\theta_1 - + \beta \sin(2(\theta_1 - \theta_2) \quad (7)$$

$$\dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial \theta_2} = -m_2 g l_2 \sin\theta_2 + \alpha - \beta \sin\big(2(\theta_1 - \theta_2)\big) \quad (8)$$

Let us $\alpha = \frac{p_1 p_2 \sin(\theta_1 - \theta_2)}{l_1 l_2 \left(m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right)}$ and $\beta = \frac{m_2 l_2^2 p_1^2 + (m_1 + m_2) l_1^2 p_2^2 - 2 m_2 l_1 l_2 p_1 p_2 \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))^2}$.

To assess energy conservation, the Hamiltonian $\mathcal{H}(\theta, p)$ is monitored along the numerical trajectory. At each time step t_n, the numerical energy is computed as $\mathcal{H}_n = \mathcal{H}(q_n, p_n)$. The energy deviation is then defined with respect to the initial value as [11]

$$\Delta\mathcal{H}_n = \mathcal{H}_n - \mathcal{H}_0 \quad (9)$$

The performance of each integrator is evaluated based on the long-time behavior of this quantity, particularly the maximum deviation $\max_n |\Delta\mathcal{H}_n|$ and the presence or absence of systematic energy drift.

Numerical Integrators

To systematically evaluate the numerical behavior of the double pendulum, we consider both non-symplectic and symplectic integrators, encompassing explicit and implicit schemes. The RK4 method represents a classical explicit non-symplectic integrator. Störmer–Verlet, implicit midpoint, and Gauss–Legendre (GL2) belong to the class of symplectic integrators; among the latter, Störmer–Verlet is explicit, whereas implicit midpoint and GL2 are implicit methods. This selection enables a direct comparison between general computational accuracy and structure-preserving properties in long-term simulations. Specifically, to advance the state vector $y_n = (\theta_1, \theta_2, p_1, p_2)$ to y_{n+1} over a time step h, the following four numerical schemes are evaluated:

1. Classical Fourth Order Runge – Kutta (RK4)
   Serving as the conventional explicit baseline, the RK4 method advances the state by evaluating the right-hand side function $f(y_n)$ at four intermediate stages [1]:

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f\left(y_n + \frac{h}{2}k_1\right) \\ k_3 &= f\left(y_n + \frac{h}{2}k_2\right) \\ k_4 &= f(y_n + hk_3) \\ y_{n+1} &= y_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \tag{10}$$

   Despite its high local accuracy, RK4 does not preserve the symplectic structure, which may lead to long-term energy drift.
2. Störmer–Verlet
   The Störmer–Verlet scheme is a second-order symplectic integrator particularly suited for Hamiltonian systems. Its one-step formulation reads

$$\begin{aligned} p_{n+\frac{1}{2}} &= p_n - \frac{h}{2}\nabla_\theta V(\theta_n) \\ \theta_{n+1} &= \theta_n + hp_{n+\frac{1}{2}} \\ p_{n+1} &= p_{n+\frac{1}{2}} - \tfrac{h}{2}\nabla_\theta V(\theta_{n+1}) \end{aligned} \tag{11}$$

   This method preserves the geometric structure of phase space and exhibits bounded energy error over long time integration [12].
3. Implicit Midpoint
   This symplectic method evaluates the rates of change exactly at the midpoint of the integration interval [13]. In canonical coordinates, this becomes

$$\begin{aligned} \theta_{n+1} &= \theta_n + h\frac{\partial \mathcal{H}}{\partial p}\left(\frac{\theta_n + \theta_{n+1}}{2}, \frac{p_n + p_{n+1}}{2}\right) \\ p_{n+1} &= p_n - h\frac{\partial \mathcal{H}}{\partial \theta}\left(\frac{\theta_n + \theta_{n+1}}{2}, \frac{p_n + p_{n+1}}{2}\right) \end{aligned} \tag{12}$$

   The resulting nonlinear system is solved iteratively at each time step.
4. Gauss–Legendre Runge–Kutta (GL2)
   The two-stage Gauss–Legendre method (GL2) is a fourth-order implicit symplectic Runge–Kutta scheme. It is defined as [14]

$$y_{n+1} = y_n + h\sum_{i=1}^{2} b_i k_i\,, \qquad b_1 = b_2 = \frac{1}{2} \tag{13}$$

with stage equations [15]

$$k_i = f\left(y_n + h\sum_{j=1}^{2} a_{ij}k_j\right) \tag{14}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{bmatrix}$.

For implicit schemes, the resulting nonlinear systems are solved using fixed-point iteration initialized with an explicit Euler predictor. Convergence is assessed using the Euclidean norm of successive iterates, i.e., $\|y_{n+1} - y_n\| < 10^{-12}$ or until a maximum of 100 iterations is reached.

Implementation Details

In all numerical experiments, we consider a planar double pendulum with the physical quantities and units summarized in Table 1 [16].

Table 1. Physical quantities and units

	quantities	units
Mass 1	1	kg
Mass 2	1	kg
Length 1	1	m
Length 2	1	m
Initial angle 1	45	degrees
Initial angle 2	-90	degrees
Initial momentum 1	0	kg . m/s
Initial momentum 2	0	kg . m/s
g	9.81	m/s²

All numerical simulations were implemented in Python using TensorFlow, with double-precision floating-point arithmetic (float64) to mitigate round-off errors in long-term integration. The system state is represented as $y_n = (\theta_{1,n}, \theta_{2,n}, p_{1,n}, p_{2,n})^{\mathsf{T}}$, and the vector field is evaluated directly from Hamilton's equations. A uniform time step $h = 10^{-3}$ is employed over a fixed simulation horizon $[0\,;\,100\,s]$, resulting in $N = 10^5$ integration steps per trajectory [17]. This configuration is kept identical across all numerical methods to ensure a fair comparison.

For implicit integrators (implicit midpoint and GL2), the nonlinear stage equations are solved using fixed-point iteration with an explicit Euler initialization. A maximum of 100 iterations is performed per time step, with convergence monitored using a prescribed tolerance. At each time step, the Hamiltonian $\mathcal{H}(\theta_1, \theta_2, p_1, p_2)$is evaluated and recorded for subsequent analysis of energy conservation. All experiments were conducted using a consistent computational environment (Google Colab), ensuring reproducibility of the results.

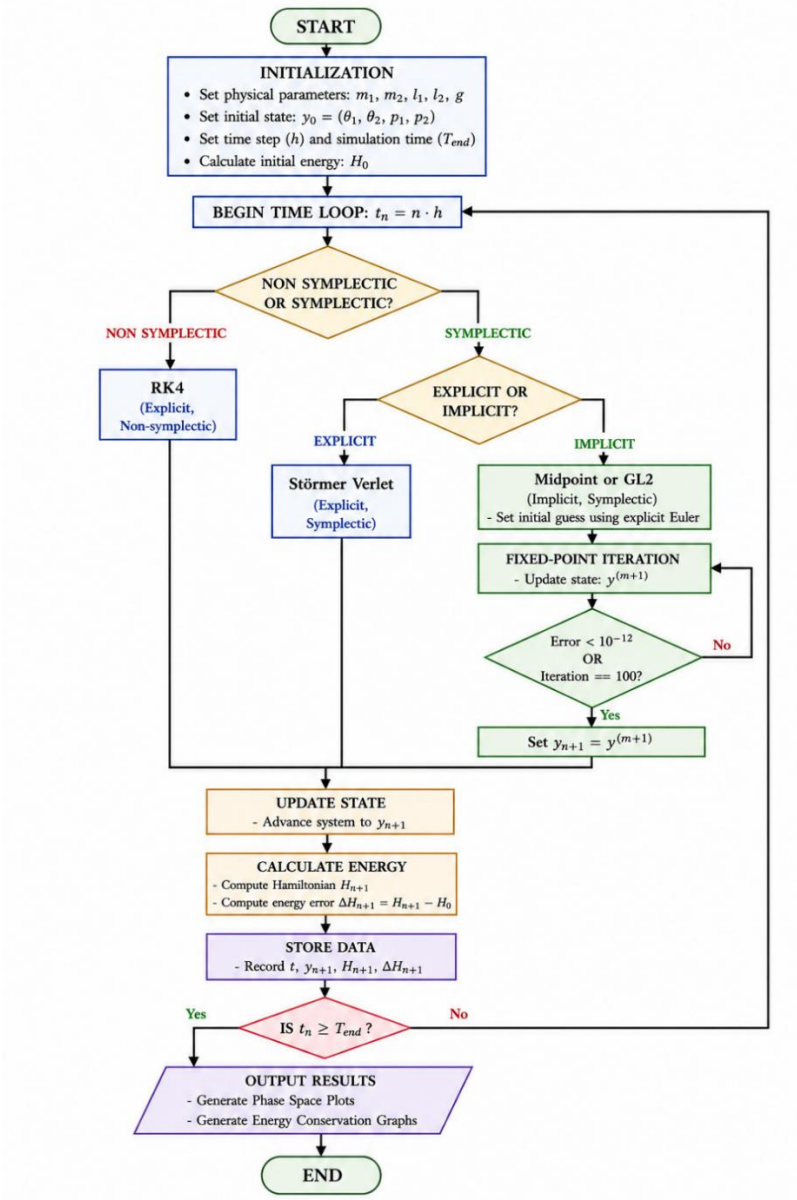


Figure 2. Flowchart of the numerical simulation procedure for the double pendulum

RESULTS AND DISCUSSION

This section presents a comparative evaluation of four numerical integrators applied to the planar double pendulum Hamiltonian system, namely RK4, Störmer–Verlet, implicit midpoint, and GL2. All simulations are performed using identical physical parameters (Table 1), time-step size, and integration horizon to ensure a consistent basis for comparison. The analysis focuses on three key aspects: qualitative phase-space behavior, long-time energy conservation, and computational efficiency.

Phase-Space Analysis

Figure 3 illustrates the phase-space trajectories in the (θ_1, p_1) and (θ_2, p_2) planes generated by the four integrators. The implicit symplectic integrators (Implicit Midpoint and GL2) successfully preserve the geometric structure of the phase space, displaying well-defined, bounded orbital

regions characteristic of stable Hamiltonian dynamics. The RK4 method also produces visually bounded trajectories over this simulation period; however, its non-symplectic nature typically leads to mild deformations and gradual deviation from the exact energy manifold, which becomes more apparent in long-term energy analysis.

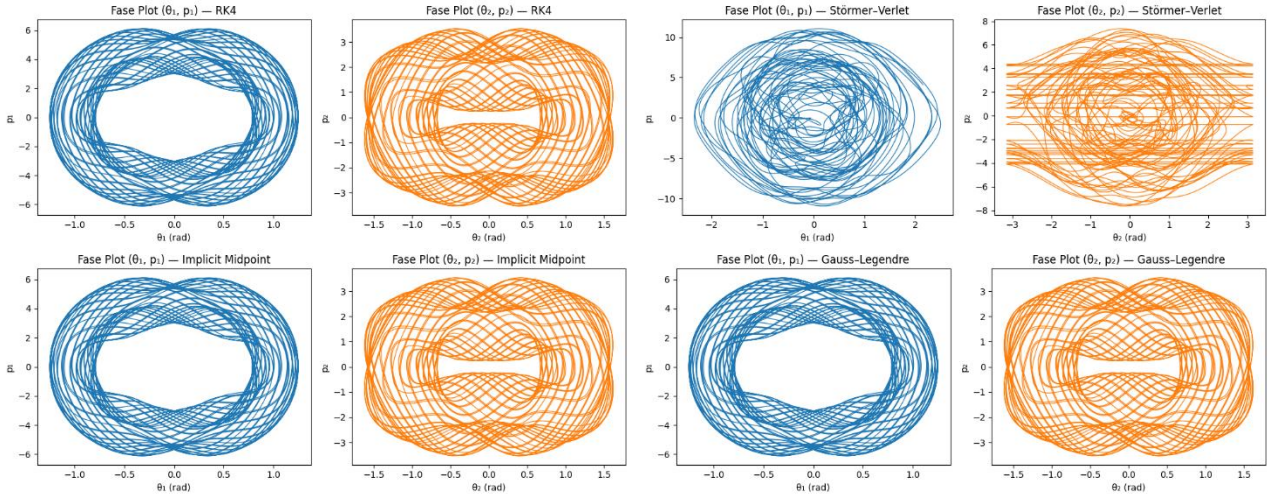


Figure 3. Phase plot in the (θ, p) plane of the numerical solution using RK4 (top left), Störmer–Verlet (top right), Implicit Midpoint (bottom left), and GL2 (bottom right) integrators

By contrast, the Störmer–Verlet method shows severe degradation of qualitative accuracy, with trajectories scattering chaotically and extending far beyond the typical physical bounds observed in the reference solutions [16]. A robust explanation for this unexpected behavior is that the standard explicit Störmer–Verlet formulation relies on the assumption of a separable Hamiltonian ($\mathcal{H}(q,p) = T(p) + V(q)$). The double pendulum, however, possesses a strongly coupled, non-separable Hamiltonian where the kinetic energy depends heavily on the angular coordinates. The present implementation uses the standard separable Störmer–Verlet formulation without specialized splitting for non-separable Hamiltonians. Applying the standard explicit scheme to such a coupled system violates the underlying structural assumptions of the integrator, leading to numerical instability. Consequently, due to this fundamental incompatibility, the explicit Störmer–Verlet method is excluded from the subsequent detailed energy conservation analysis.

The observed instability should not be interpreted as a general failure of all Verlet-type methods for non-separable Hamiltonians. Rather, it reflects the limitation of the standard explicit Störmer–Verlet formulation used in this implementation. More advanced splitting or variational symplectic formulations may provide improved stability for strongly coupled systems and remain an important direction for future work. Overall, the comparison indicates that symplectic integrators, particularly implicit midpoint and GL2, better preserve the geometric structure of the phase space, whereas RK4 exhibits mild deformation consistent with a gradual deviation from a single energy level.

Energy Conservation

The long-time evolution of the Hamiltonian provides a quantitative measure of structure preservation. Figures 4 show the energy deviation $\Delta\mathcal{H}(t) = \mathcal{H}(t) - \mathcal{H}_0$ for the RK4, implicit midpoint, and GL2 methods.

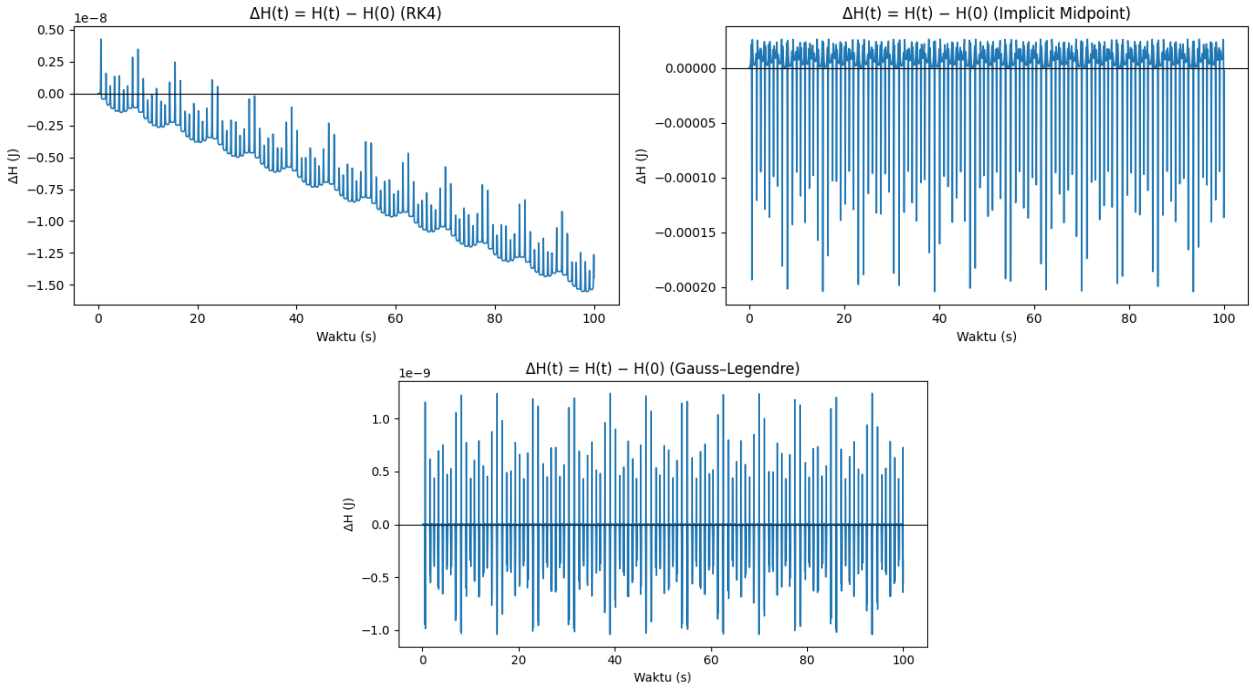


Figure 4. Relative energy error ($\Delta\mathcal{H}_n$) over time for the RK4 (top left), Implicit Midpoint (top right), and GL2 (bottom) integrators

The RK4 scheme exhibits small-amplitude oscillations superimposed on a clear secular drift, with the energy gradually deviating from its initial value. This behavior is consistent with the non-symplectic nature of classical RK4 methods, which do not preserve the underlying geometric structure of Hamiltonian systems. In contrast, the implicit midpoint method shows bounded, oscillatory energy error with no visible long-term drift. Although the instantaneous deviations are larger in magnitude, they remain confined within a fixed envelope and change sign over time. This pattern is characteristic of symplectic integrators, which preserve a modified Hamiltonian and prevent systematic energy accumulation.

Among the tested methods, the GL2 integrator demonstrates the best energy conservation. The energy deviation remains tightly bounded around zero, with very small oscillations and no observable drift over the entire simulation interval. This superior behavior is consistent with the theoretical properties of GL2 collocation methods, which combine high-order accuracy with symplectic structure preservation.

Quantitative Energy Metrics

To complement the qualitative energy plots in Figure 4, Table 2 reports quantitative measures of energy conservation, including the maximum absolute deviation max | ΔH | and the root-mean-square (RMS) error.

Table 2. Quantitative Energy Metrics

Method	$\mathcal{H}_0(J)$	Max $\Delta\mathcal{H}(J)$	RMS $\Delta\mathcal{H}(J)$
RK4	-13.87	1.55×10^{-8}	8.83×10^{-9}
Implicit Midpoint	-13.87	2.04×10^{-4}	3.24×10^{-5}
GL2	-13.87	1.24×10^{-9}	2.05×10^{-10}

The RK4 method achieves relatively small instantaneous errors but exhibits a systematic drift over time. The implicit midpoint method produces significantly larger instantaneous deviations, reflecting its oscillatory energy behavior. In contrast, the GL2 integrator achieves the lowest values across both metrics, confirming its superior ability to preserve the Hamiltonian both locally and globally.

Sensitivity to Time-Step Size

To assess robustness, additional simulations were conducted using varying time-step sizes $h = 10^{-1}, 10^{-2}, 10^{-3}$. Figure 5 shows that all methods improve in accuracy as h decreases, consistent with their theoretical orders.:

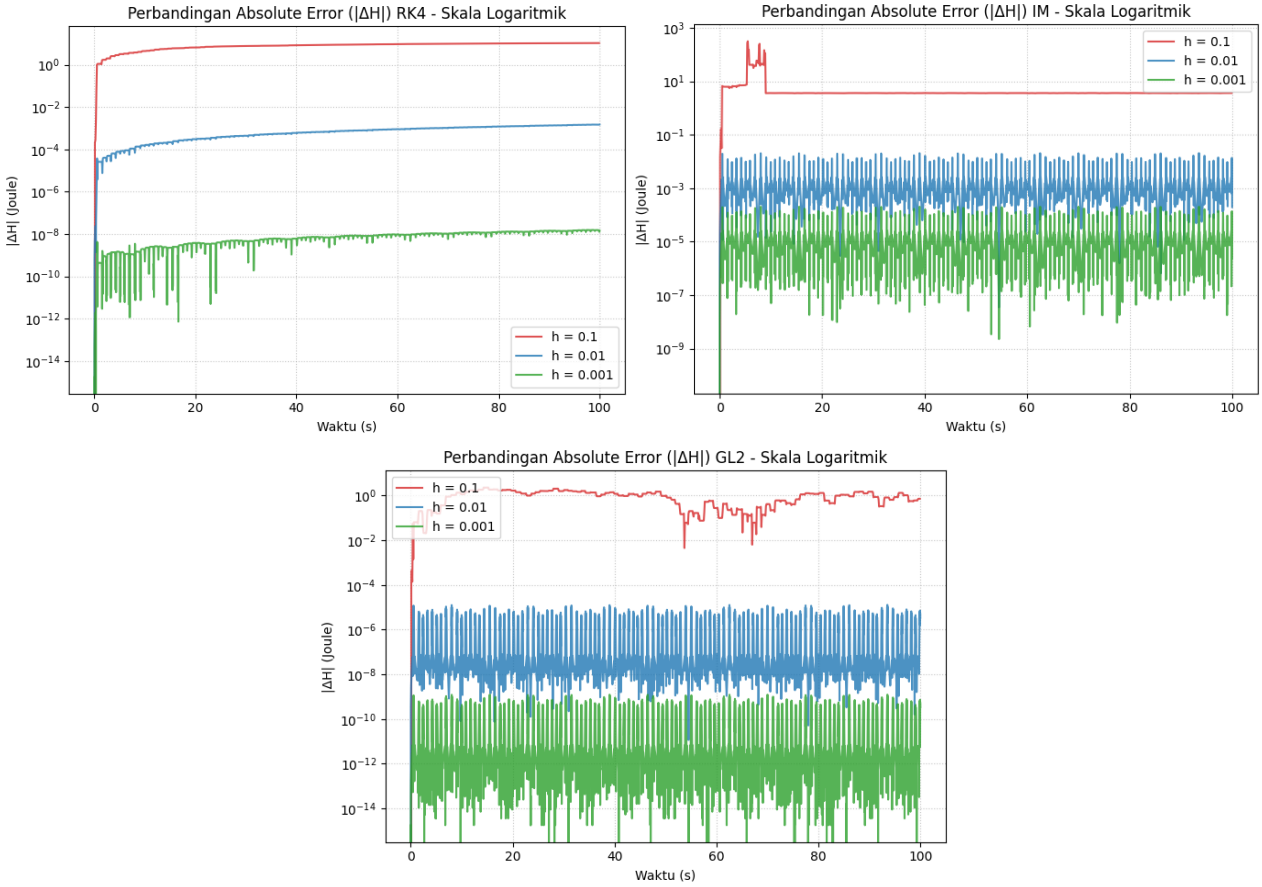


Figure 5. Time-step sensitivity analysis comparing the absolute Hamiltonian error ($\Delta\mathcal{H}_n$)of the RK4 (top left), Implicit Midpoint (top right), and GL2 (bottom) integrators methods using h = 0.1, 0.01, and 0.001 s.

However, qualitative differences persist. The RK4 method exhibits increasing energy drift for larger step sizes, indicating sensitivity to discretization. In contrast, symplectic integrators maintain bounded energy errors even for relatively coarse time steps. GL2 integrator, in particular, demonstrates strong robustness, maintaining both small error magnitude and stability across all tested values of h.

These observations confirm that the superior performance of symplectic methods is not limited to a specific parameter choice but reflects intrinsic structural properties. To further quantify this behavior, an empirical convergence analysis is presented in the following subsection.

Empirical Convergence Analysis

To further assess the numerical accuracy of the tested integrators, an empirical convergence analysis was performed by varying the time-step size h. The final Hamiltonian error was evaluated at the end of the simulation interval and defined as:

$$E_{final}(h) = |H(q_N, p_N) - H(q_0, p_0)| \quad (15)$$

The simulations were conducted using three time-step sizes, namely $h = 10^{-1}, 10^{-2}$, and 10^{-3} s, while all physical parameters and initial conditions were kept fixed. The empirical order of convergence was then estimated from the slope of the log-log relation between the final energy error and the time-step size:

$$p \approx \frac{log\left(\frac{E(h1)}{E(h2)}\right)}{log\left(\frac{h1}{h2}\right)} \quad (16)$$

A smaller final Hamiltonian error indicates better long-time energy accuracy. For sufficiently small h, the expected convergence behavior is approximately second order for the implicit midpoint method and fourth order for RK4 and GL2.

Table 3. Final Hamiltonian Error for Different Time-Step Sizes

Method	$h = 10^{-1}$ (J)	$h = 10^{-2}$(J)	$h = 10^{-3}$(J)	Estimated Order
RK4	1.05×10^{1}	1.48×10^{-3}	1.44×10^{-8}	≈ 4
Implicit Midpoint	3.61×10^{0}	1.96×10^{-4}	2.42×10^{-6}	≈ 2
GL2	7.24×10^{-1}	3.35×10^{-6}	3.38×10^{-10}	≈ 4

For sufficiently small time steps, the implicit midpoint method exhibits an approximately second-order trend, while RK4 and GL2 demonstrate steeper error reduction consistent with fourth-order accuracy. A more detailed examination indicates that the estimated convergence order is more reliable when computed using smaller step sizes. In particular, the implicit midpoint method shows a convergence rate close to second order when comparing $h = 10^{-2}$ and $h = 10^{-3}$, while both RK4 and GL2 approach fourth-order behavior in this refined regime. In contrast, estimates based on larger step sizes tend to overestimate the convergence rate, indicating that the asymptotic regime has not yet been fully reached.

Despite the similar formal order of RK4 and GL2, their long-time energy behavior differs substantially. RK4 exhibits significantly larger energy errors at coarse step sizes and is prone to systematic drift due to its non-symplectic nature. In contrast, GL2 maintains consistently smaller final Hamiltonian errors across all tested step sizes, reflecting its symplectic structure and improved long-time stability.

Computational cost and efficiency

To complement the accuracy and energy–preservation analysis, we also compare the computational cost of the three integrators. Table 4 summarizes the computational cost of each method for $N = 10^5$ time steps.

Table 4. Computational cost and energy drift for the three integrators

Method	Time (s)	Steps/s
RK4	4199.60	24
Implicit Midpoint	3089.03	32
GL2	590.80	169

Despite being an implicit method, the GL2 integrator achieves the highest computational efficiency in this implementation. This is likely due to stable convergence of the fixed-point iteration and efficient vectorized computation. The implicit midpoint method is moderately faster than RK4 but exhibits less favorable accuracy. The RK4 method is the slowest and additionally suffers from energy drift, making it less suitable for long-time simulations. However, the unexpectedly high throughput of GL2 is implementation-dependent and likely influenced by TensorFlow vectorization and stable fixed-point convergence. Therefore, the runtime comparison should not be interpreted as a universal theoretical advantage of GL2 over explicit RK4 methods.

Discussion

The results consistently demonstrate that the choice of numerical integrator significantly affects both qualitative and quantitative properties of Hamiltonian simulations. Symplectic methods preserve the geometric structure of phase space and maintain bounded long-time energy behavior, whereas non-symplectic methods such as RK4, despite achieving smaller instantaneous energy errors, may exhibit gradual secular drift over extended simulations. The empirical convergence analysis further confirms that the numerical error decreases as the time-step size is refined, consistent with the theoretical convergence behavior of the tested methods.

Among the tested schemes, the GL2 integrator provides the best overall performance, combining high-order accuracy, excellent energy conservation, consistently smaller final Hamiltonian errors, improved long-time stability, and strong computational efficiency. Although RK4 and GL2 both exhibit approximately fourth-order convergence, GL2 maintains more stable long-time energy behavior due to its symplectic structure. In particular, RK4 achieves very small instantaneous energy errors due to its high local accuracy, as reflected in the quantitative metrics. However, these errors accumulate gradually over long simulations, producing secular energy drift. By contrast, the implicit midpoint method exhibits larger instantaneous energy deviations but maintains bounded oscillatory energy behavior, which is characteristic of symplectic integrators. The implicit midpoint method, while symplectic, therefore reflects a trade-off between long-time stability and pointwise accuracy..

The unexpected behavior of the Störmer–Verlet method highlights the importance of method-specific assumptions and parameter sensitivity. Further investigation is needed to determine whether this issue arises from step-size selection, formulation details, or the inherent coupling structure of the system.

CONCLUSION

This study systematically evaluates the impact of numerical integration schemes on the long-time simulation of a planar double pendulum formulated in Hamiltonian form. By combining qualitative phase-space analysis, quantitative energy metrics, empirical convergence analysis, and

computational efficiency benchmarks under a unified experimental framework, the results demonstrate that structure-preserving (symplectic) integrators provide fundamentally more reliable behavior than conventional methods for chaotic Hamiltonian systems.

The main contribution of this work lies in establishing a comparative and reproducible benchmark that highlights the trade-off between accuracy, stability, and computational cost across both symplectic and non-symplectic schemes. In particular, the Gauss–Legendre (GL2) integrator consistently achieves the best overall performance, exhibiting minimal energy deviation, strong robustness to time-step variation, and superior computational efficiency within the tested configuration. The convergence analysis further shows that although RK4 and GL2 both achieve approximately fourth-order accuracy, GL2 maintains more stable long-time energy behavior due to its symplectic structure. While RK4 attains high local accuracy, its non-symplectic nature leads to systematic energy drift, whereas the implicit midpoint method preserves qualitative structure but exhibits larger instantaneous energy oscillations.

An additional contribution of this study is the clarification of the limitations of the explicit Störmer–Verlet method when applied to non-separable Hamiltonian systems such as the double pendulum. The observed instability is attributed to a fundamental mismatch between the method's structural assumptions and the strongly coupled nature of the system, providing insight into the appropriate applicability of symplectic schemes.

Despite these findings, the present study remains limited to a specific system configuration, a restricted range of time-step sizes, and a fixed implementation of the numerical methods. Future work should include a more rigorous convergence and error-scaling analysis, a systematic investigation of step-size sensitivity (particularly for Störmer–Verlet variants), and the extension to higher-order symplectic integrators. Furthermore, the proposed benchmarking framework can be extended to evaluate data-driven structure-preserving models, such as Hamiltonian Neural Networks, thereby bridging classical geometric integration with modern machine learning approaches.

In summary, this work reinforces the importance of structure preservation in long-time Hamiltonian simulation and provides a practical reference for selecting appropriate integrators in both physics-based and data-driven dynamical systems.

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AUTHOR CONTRIBUTIONS

Hilman Nuha Ramadhan : Conceptualization, Methodology, Formal Analysis, Resources, Project Administration, and Writing - Original Draft; and Novitrian : Methodology, Validation, and Supervision

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